On Equilibrium Existence in a Finite-Agent, Multi-Asset Noisy Rational Expectations Economy

Ronaldo Carpio*[1] and Meixin Guo†[2]

[1] *School of Banking & Finance, University of International Business & Economics*

[2] *Department of Economics, Tsinghua University*

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Abstract

In this paper we introduce a novel method of proving existence of rational expectations equilibria (REE) in multi-dimensional CARA-Gaussian environments. Our approach is to focus on the set of parameters characterizing agents' beliefs. We show that this set is convex and compact; we then construct a continuous mapping from agents' initial beliefs, to optimal behavior given these beliefs, to equilibrium behavior; and then finally, to agents' updated beliefs after observing equilibrium. We appeal to Brouwer's fixed point theorem to prove that a fixed point exists, which must be a REE. We use this method to to prove existence of REE in a finite-agent version of the model of Admati (1985), which is a multi-asset noisy REE asset pricing model with dispersed information. Our method can prove existence in models that are not currently handled by the literature; it also can be applied to any multi-dimensional REE model with Gaussian uncertainty and behavior that is linear in agents' information.

Keywords: asymmetric information, noisy rational expectations, multiple assets, equilibrium existence

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*Corresponding author. 9th Floor, Boxue Bldg., 10 Huixindongjie, Beijing, 100029, China. Email: rncarpio@yahoo.com. Tel: +86 18811577940, Fax: +86 1064495059.

†Department of Economics, School of Economics and Management, Tsinghua University, Beijing, China, 100084. Email: guomx@sem.tsinghua.edu.cn. Tel: (8610) 6279-5839.

1 Introduction

In this paper we introduce a novel method of proving existence of rational expectations equilibria (REE) in multi-dimensional CARA-Gaussian environments, and use it to prove existence of REE in a finite-agent version of the model of Admati (1985), which is a multi-asset noisy REE model with dispersed information among many agents.

In such a model, agents' Gaussian beliefs are characterized by a mean vector and a positive semidefinite covariance matrix. The covariance matrix encodes all linear relationships between random variables; if the covariance matrix can be determined, then it, together with the law of iterated expectations, will pin down the mean vector under rational expectations. First, we show that the set of valid belief covariance matrices is convex and compact. Then, we construct a mapping from agents' initial beliefs, to optimal behavior, to equilibrium; and then to agents' updated beliefs, after observing equilibrium. We show that this mapping is continuous and takes the set of covariance matrices into itself; by Brouwer's fixed point theorem, a fixed point exists, which must be a REE. Two properties of Gaussians are key: first, the set of positive semidefinite matrices is a convex subset of Euclidean space. Second, the operations involved in the mapping (conditioning on a subset of variables, and taking a linear transformation of variables) preserves the Gaussian property. Under CARA (and mean-variance) utility, each agent's optimal demand is linear in their information plus prices, which implies that equilibrium prices are also linear in all agents' information. This guarantees that agents' updated beliefs remain Gaussian, and thus the mapping takes a Gaussian to another Gaussian.

This paper fills a gap in the noisy REE literature for models with dispersed information. Existence of REE in the finite-agent, one-risky-asset case was proven in Hellwig (1980) (see Lemma 3.1) under Gaussian uncertainty, and in Breon-Drish (2015) (see Lemma A15) under exponential family uncertainty. These proofs applied Brouwer's fixed point theorem to one-dimensional intervals. However, as noted in Admati (1985) (see the discussion preceding footnote 8), this approach does not carry over to the multi-asset case, since it depends on properties of scalars which do not carry over to matrices. Furthermore, while the proofs mentioned above use constructions specific to the models in question, our method is more general, and can be applied to any model where agents have Gaussian beliefs. To numerically solve for the REE, well-known iterative algorithms such as Mann iteration can be used to compute a fixed point of the mapping. Our approach relies heavily on the properties of positive semidefinite matrices. While this increases the technical burden on the reader, we believe that this is the natural way to extend results from one-dimensional Gaussian models to higher dimensions.

The papers mentioned above also consider a model with an infinite number of agents, and are able to prove existence of a linear REE by deriving the coefficients of the equilibrium price function in closed form. The difference between finite and infinite-agent models may seem like a technical detail, but it can have important consequences for behavior. In infinite-agent noisy REE models, it is assumed that each agent receives an i.i.d. signal correlated with the unobserved asset payoff, and that a law of large numbers can be applied when computing the *realized* average signal across agents (e.g. Eqn. 16 in Admati (1985)). This allows modelers to equate the *realized* average belief mean across agents with the *expectation* of the belief mean distribution, which makes it possible to find an analytic solution. However, this assumption has two effects: first, the infinite i.i.d signals (and therefore the market price) contain enough information to perfectly reveal the asset payoff, removing the incentive to trade. In order to make prices only partially revealing, an additional source of noise must be added, typically to asset supply, which is commonly interpreted as "noise traders". In this paper, we retain the noisy supply assumption to facilitate the proof; however, in a finite-agent setting, this assumption is not necessary to prevent perfect revelation of payoffs, since even an omniscient agent who observes all private signals will retain some posterior uncertainty. Second, the average belief mean across agents is a constant, a.s. This means that there cannot be any meaningful effects that depend on the realized distribution of belief means among agents after they observe their private signals. Carpio and Guo (2018) gives an example of one such effect, the "diversification discount", which is the phenomenon that a conglomerate firm with multiple business segments seems to receive a lower market value than a similar collection of single-segment firms[1]. In that paper we show that under the infinite-agent assumption, a discount exists a.s., but with finite agents, a discount or a premium may exist. Since diversification premia are sometimes observed empirically (Villalonga 2004), a theory that only allows a discount would seem to suffer a drawback.

1.1 Related Literature

The seminal papers of Grossman and Stiglitz (1980) and Hellwig (1980) presented models of financial markets with a noisy rational expectations equilibrium. In these models, agents trade assets with

[1] The mechanism in this paper is essentially similar to the "trade-restriction" channel in Dai (2018)

unobserved Gaussian payoffs; "informed" agents are endowed with a private signal that reveals some information about the true payoff. Prices not only clear the market, but also allow agents to infer the demands (and hence the private signals) of other agents. In Grossman and Stiglitz (1980), there are two types of agents with a "hierarchical" information structure, in that the information set of the "informed" investors contains everything in the information set of the "uninformed" investors. This simplifies the problem, since the market price can be ignored when determining the informed agents' asset holdings. Hellwig (1980) introduced an economy with a single risky asset and many agents with a "dispersed" or "differential" information structure, where each agent has some information that no other agent knows; this paper noted the "schizophrenic" nature of agents in a finite market who are aware of the covariance between prices and their own signals and actions, but behave as price-takers. Admati (1985) extended Hellwig (1980) to include an arbitrary number of risky assets; Breon-Drish (2015) extended both Grossman and Stiglitz (1980) and Hellwig (1980) by allowing uncertainty to follow a one-dimensional exponential family, which includes the Gaussian distribution as a special case.

This paper is related to the literature studying existence and uniqueness of equilibria in noisy rational expectations models. Pálvölgyi and Venter (2015a) show that in the model of Hellwig (1980) with one risky asset and infinite agents, there is a unique linear equilibrium and many discontinuous equilibria. Pálvölgyi and Venter (2015b) show that in a multi-asset model with infinite agents and a hierarchical information structure, the linear equilibrium is unique (if it exists). Chabakauri, Yuan, and Zachariadis (2016) model a contingent-claims market with a finite but arbitrary number of states, and study complete and incomplete market settings with asymmetric information. The noisy REE model has also been extended to a dynamic setting. He and Wang (1995) study a finite-horizon model with differential information and infinite agents. Zhou (1998) studies a special case of dispersed information with finite agents, in which there are two agents and two risky assets $i = 1, 2$, and each agent i is perfectly informed about one asset, but uninformed about the other.

The remainder of the paper is organized as follows. Section 2 presents mathematical preliminaries and standard results on positive definite matrices and Gaussian distributions. Section 3 presents a finite-agent version of the multi-asset model of Admati (1985). Section 3.1 defines the domain and continuous mapping and shows that the conditions for Brouwer's theorem apply. Section 5 concludes.

2 Preliminaries

Let $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$) denote the set of $n \times n$-dimensional, real-valued, symmetric, positive semidefinite (positive definite) matrices. $\mathbb{S}^n_+$ is a closed, unbounded, convex cone in $\mathbb{R}^{n^2}$ whose boundary rays are the rank-1 matrices.

The following results are well known.

Definition 2.1 (Loewner partial order)**.** For $A, B \in \mathbb{S}^n_+$, $A \geqslant_L B$ whenever $(A - B) \in \mathbb{S}^n_+$, and $A >_L B$ whenever $(A - B) \in \mathbb{S}^n_{++}$. Let $A \geqslant_L 0$ ($A >_L 0$) denote that A is positive semidefinite (positive definite).

The Loewner ordering has the following statistical interpretation (Horn (1990), p.141): suppose $\tilde{X}, \tilde{Y}$ are $\mathbb{R}^n$-valued random variables, with $Var(\tilde{X}) = A$ and $Var(\tilde{Y}) = B$. Then $A \geqslant_L (>_L) B$ iff for any nonzero $c \in \mathbb{R}^n$, $Var(c \cdot \tilde{X}) \geqslant (>) Var(c \cdot \tilde{Y})$. Furthermore, if $\tilde{X}, \tilde{Y}$ are Gaussian, then $Var(\tilde{X}|\tilde{Y}) \leqslant_L Var(\tilde{X})$ (see Lemma 6.5), i.e. uncertainty must weakly decrease after conditioning on an event. Thus, it is the natural counterpart to the standard ordering of the reals when dealing with variances of many variables. We can treat $\leqslant_L$ much like $\leqslant$ when defining convex regions: for fixed $A, B \in \mathbb{S}^n_+$, the set $\{X \in \mathbb{S}^n_+ | A \leqslant_L X \leqslant_L B\}$ is convex and compact. Under the Loewner ordering, we can also define conditions for concavity and monotoncity, analogous to those for real-valued functions, that apply to matrix-valued functions.

Lemma 2.1 (Conditional distribution of Gaussians)**.** *Suppose $\tilde{X}, \tilde{Y}$ are respectively m, n-dimensional jointly Gaussian random vectors with variance $M \in \mathbb{S}^{m+n}_+$, partitioned as*

$$M = \begin{bmatrix} Var(\tilde{X}) & Cov(\tilde{X}, \tilde{Y}) \\ Cov(\tilde{Y}, \tilde{X}) & Var(\tilde{Y}) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \tag{2.1}$$

Then the conditional distribution of $\tilde{X}$ given $\tilde{Y}$ is Gaussian, with mean and variance (Schott (2017), Example 7.4):

$$E[\tilde{X}|\tilde{Y}] = E[\tilde{X}] + M_{12} M_{22}^{-} \left(\tilde{Y} - E[\tilde{Y}] \right) \tag{2.2}$$

$$Var(\tilde{X}|\tilde{Y}) = M_{11} - M_{12} M_{22}^{-} M_{12}^T \tag{2.3}$$

where M_{22}^{-} denotes a generalized inverse of M_{22}. If M_{22} is nonsingular, this is identical to the standard *matrix inverse.*

Corollary 2.1 (Continuity of conditional Gaussian distribution)**.** *The mappings $M \to M_{12}M_{22}^{-1}$ (i.e. the coefficient of $\tilde{Y}$ in $E[\tilde{X}|\tilde{Y}]$) and $M \to M_{11} - M_{12}M_{22}^{-1}M_{12}^T = Var(\tilde{X}|\tilde{Y})$ are continuous over $\{M \in \mathbb{S}_+^{m+n} | M_{22} \in \mathbb{S}_{++}^n\}$.*

When dealing with a one-dimensional variance parameter, we frequently want to define a set of possible variances that is bounded and bounded away from zero; this ensures closedness and boundedness, while avoiding the degenerate case of a zero variance. The corresponding degeneracy condition for a matrix is singularity, which occurs if there is perfect collinearity among the set of random variables (equivalently, the unexplained variance after conditioning some on subset of variables is zero). Later in the paper we will want to define a closed subset of $\mathbb{S}_+^n$ for our mapping to operate on; however, this subset cannot contain a singular matrix, since our mapping will include the matrix inverse, which will be undefined.[2] We will refer to a subset of $\mathbb{S}_+^n$ as being "bounded" or being "bounded away from singular" in the Loewner ordering; the following results give equivalencies for these conditions.

Lemma 2.2 (Bounded positive semidefinite matrices)**.** *Suppose $A \in \mathbb{S}_+^n$. The following statements are equivalent:*

(a) A is bounded in the Loewner order: there exists $B \in \mathbb{S}_+^n$ such that $A \leqslant_L B$.

(b) A is bounded element-wise.

(c) The maximum eigenvalue of A is bounded.

Lemma 2.3 (Positive definite matrices bounded away from singular)**.** *Let $M = Var(\tilde{X}, \tilde{Y})$, and suppose $M \in \mathbb{S}_{++}^{m+n}$ and is partitioned as in Lemma 2.1. The following statements are equivalent:*

(a) M is bounded away from singular in the Loewner order: there exists $B \in \mathbb{S}_{++}^{m+n}$ such that $M \geqslant_L B$.

(b) The minimum eigenvalue of M is bounded away from zero.

(c) Both $Var(\tilde{X}|\tilde{Y})$ and $Var(\tilde{Y}|\tilde{X})$ are bounded away from singular in the Loewner order.

Corollary 2.2 (Inverse of bounded matrix is bounded away from singular)**.** *If $A \in \mathbb{S}_{++}^n$, then A is bounded iff A^{-1} is bounded away from singular.*

[2] We could replace the matrix inverse with a pseudoinverse; however, continuity of a pseudoinverse is not guaranteed at a singular matrix.

2.1 Convexity and Compactness of a Set of Covariance Matrices with Constraints

In order to apply Brouwer's fixed point theorem, we need to show that the set of covariance matrices characterizing agents' beliefs is convex and compact. Here we present a general result for a set of possible covariance matrices of a Gaussian random vector, subject to two types of constraints: (i) upper and lower bounds on the joint variance of some subset of variables; (ii) a lower bound on the *conditional* variance of some subset of variables, given another subset. As we will see, in Admati's model, the set of possible agent beliefs will be defined by constraints of these two types.

Theorem 2.1 (Convexity of set of covariance matrices subject to constraints)**.** *Suppose $\tilde{X} = (\tilde{x}_1, ...\tilde{x}_n)$ is an n-dimensional, jointly Gaussian random vector. We impose two types of constraints on $Var(\tilde{X})$:*

(a) Let $\tilde{X}_I = (\tilde{x}_{i_1}, ..., \tilde{x}_{i_I})$ denote a subset of $\tilde{X}$ of length I, and let A, B be fixed matrices in $\mathbb{S}^I_+$. Suppose we impose a constraint of the form $A \leqslant_L Var(\tilde{X}_I) \leqslant_L B$. The set of $Var(\tilde{X}_I)$ that satisfies this constraint is a convex, compact subset of $\mathbb{S}^I_+$. This also implies that each of $Var(\tilde{x}_{i_1})$, ..., $Var(\tilde{x}_{i_I})$ is bounded.

(b) Let $\tilde{X}_I = (\tilde{x}_{i_1}, ..., \tilde{x}_{i_I})$, $\tilde{X}_J = (\tilde{x}_{j_1}, ..., \tilde{x}_{j_J})$ denote two disjoint subsets of $\tilde{X}$ of length I and J, respectively, and let A, B be fixed matrices such that $A \in \mathbb{S}^J_{++}, B \in \mathbb{S}^I_+$. Suppose we impose the following constraints: (i) $Var(\tilde{X}_J) \geqslant_L A >_L 0$; (ii) $Var(\tilde{X}_I | \tilde{X}_J) \geqslant_L B \geqslant_L 0$. That is, we impose a fixed lower bound on the conditional variance of $\tilde{X}_I$ given $\tilde{X}_J$; this requires that $Var(\tilde{X}_J)$ be nonsingular, which is guaranteed by (i). The set of $Var(\tilde{X}_I, \tilde{X}_J)$ that satisfies this constraint is a closed, convex, and unbounded subset of $\mathbb{S}^{I+J}_{++}$.

Suppose we impose any number of constraints of type (a) and type (b); then the set of $Var(\tilde{X})$ that satisfies these constraints is convex and closed. Furthermore, if as a result of conditions of type (a), each of $Var(\tilde{x}_1), ...Var(\tilde{x}_n)$ is bounded, then it is also bounded.

3 The model: Admati (1985) with finite agents

There are I agents and two periods; agents trade in the first period and consume in the second. Each agent $i = 1, ..., I$ invests his initial wealth w_{0i} in a riskless asset and N risky assets. Agent i has population mass α_i, where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. The riskless rate of return is assumed to be 1. Let $\tilde{F}$ denote the random vector of risky asset returns. Agent i's final wealth is

$$\tilde{w}_{1i} = w_{0i}R + D_i^T(\tilde{F} - \tilde{P}R) \tag{3.1}$$

D_i is the vector of holdings of risky assets, and $\tilde{P}$ is the vector of market prices, also a random variable. Each agent i has exponential utility $u_i(w) = -\exp(-w/\rho_i)$. At the beginning of period 1, each agent i receives a private signal $\tilde{Y}_i = \tilde{F} + \tilde{\epsilon}_i$, where $\tilde{\epsilon}_i$ is i.i.d. Gaussian. Let $D_i(\tilde{P}, \tilde{Y}_i)$ denote the demand vector of agent i induced by maximizing expected utility. Total asset supply is assumed to be a random variable $\tilde{Z}$.

We assume that $(\tilde{F}, \tilde{Z}, \tilde{\epsilon}_1, ..., \tilde{\epsilon}_I)$ is jointly Gaussian, with mean $(\bar{F}, \bar{Z}, 0, ..., 0)$ and block-diagonal variance matrix $diag(V, U, S_1, ..., S_I)$. All off-diagonal blocks are zero. $V, S_1, ..., S_I, U$ are positive definite. Under rational expectations, each agent i is assumed to know the true joint distribution of $(\tilde{F}, \tilde{Y}_i, \tilde{P})$. We will seek a linear rational expectations equilibrium, in which price is a linear function of private signals and asset supply:

$$\tilde{P} = A_0 + \sum_i A_{1i}\tilde{Y}_i + A_2\tilde{Z} \tag{3.2}$$

Note the difference between this and the price function in the infinite-agent model $\tilde{P} = A_0 + A_1\tilde{F} - A_2\tilde{Z}$ (Eqn 6 in Admati (1985)), in which the individual signals received by agents do not appear, only the asset payoff $\tilde{F}$. The linear form of the price function implies that $(\tilde{F}, \tilde{P}, \tilde{Z}, \tilde{Y}_1, ..., \tilde{Y}_I)$ are jointly Gaussian. Therefore, the conditional distribution of $(\tilde{F}|\tilde{Y}_i, \tilde{P})$ is also Gaussian; let $(\tilde{\mu}_i, V_i)$ denote its mean and variance, which also specifies the beliefs of agent i. Then $\tilde{\mu}_i = E[\tilde{F}|\tilde{Y}_i, \tilde{P}]$ will be a linear function of $\tilde{Y}_i$ and $\tilde{P}$; let B_{0i}, B_{1i}, B_{2i} denote its coefficients:

$$\tilde{\mu}_i = E[\tilde{F}|\tilde{Y}_i, \tilde{P}] = B_{0i} + B_{1i}\tilde{Y}_i + B_{2i}\tilde{P} \tag{3.3}$$

After standard manipulations, agent i's demand vector is given by

$$D_i(\tilde{P}, \tilde{Y}_i) = \frac{1}{\rho_i}V_i^{-1}(\tilde{\mu}_i - \tilde{P}) \tag{3.4}$$

The market clearing condition is:

$$\sum_i \alpha_i D_i(\tilde{P}, \tilde{Y}_i) = \sum_i \frac{\alpha_i}{\rho_i} V_i^{-1}(\tilde{\mu}_i - \tilde{P}) = \tilde{Z} \tag{3.5}$$

The equilibrium price vector $\tilde{P}$ must satisfy the condition

$$\tilde{P} = \left(\sum_i \frac{\alpha_i}{\rho_i} V_i^{-1}\right)^{-1} \left(\sum_i \frac{\alpha_i}{\rho_i} V_i^{-1} \tilde{\mu}_i - \tilde{Z}\right) \tag{3.6}$$

Let $W_i = \frac{\rho_i}{\alpha_i} V_i$; this is the "effective" variance of agent i's belief, incorporating the agent's risk tolerance and population mass. We can rewrite the equilibrium price condition as

$$\tilde{P} = (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}\tilde{\mu}_i - \tilde{Z}) = (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}\tilde{\mu}_i) - (\sum_i W_i^{-1})^{-1}\tilde{Z} \tag{3.7}$$

The first term in this expression, $\left(\sum_i W_i^{-1}\right)^{-1}\left(\sum_i W_i^{-1}\tilde{\mu}_i\right)$, is the Bayesian posterior mean that results from observing I normally distributed signals, where the ith signal is observed to be $\tilde{\mu}_i$ and has a known variance of W_i. The coefficient of $\tilde{Z}$ in the second term, $(\sum_i W_i^{-1})^{-1}$, is the Bayesian posterior variance. Thus, we have:

Remark 3.1 (Markets as a Bayesian aggregator of information)**.** In a linear REE, the equilibrium price is identical to an economy with a representative investor who has "observed" the beliefs of the I agents and combined them using Bayesian updating.

Let $\underline{V}$ denote $Var(\tilde{F}|\tilde{Y}_1, ..., \tilde{Y}_I, \tilde{Z}) = (V^{-1} + \sum_i S_i^{-1})^{-1}$, the variance of an agent who has observed all private signals and the noisy supply. Assuming a linear price equation of the form in Eq. 3.2 holds, and each agent's belief variance V_i is $Var(\tilde{F}|\tilde{Y}_i, \tilde{P})$, then V_i must satisfy the following two conditions, whether in a linear REE or not:

Lemma 3.1 (Boundedness of agent's belief variance)**.** *For each $i = 1, ..., I$,*

$$\underline{V} \leqslant_L V_i \leqslant_L V. \tag{3.8}$$

Lemma 3.2 (Variance of price is bounded)**.** *There exists some $\overline{V}_P \in \mathbb{S}^N_{++}$ such that $Var(\tilde{P}) \leqslant_L \overline{V}_P$.*

In a linear REE, agents' belief means must satisfy the following condition.

Lemma 3.3 (Expectation of beliefs and prices)**.** $E[\tilde{\mu}_i] = \bar{F}$ *and* $\bar{P} = E[\tilde{P}] = \bar{F} - \left(\sum_i W_i^{-1}\right)^{-1}\bar{Z}$.

3.1 Existence of Equilibrium with Finite Agents

First, we describe the mapping from agents' initial belief variances to their updated belief variances after equilibrium, and show that it is continuous. Then, we show that given the parameters of the economy, a convex, compact, and nonempty set exists, such that the mapping takes this set into itself.

3.1.1 Mapping from Initial Beliefs to Updated Beliefs

Let X_i^1 denote the set $\{Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) \in \mathbb{S}_+^{3N}\}$ subject to the following constraints:

$$Var(\tilde{F}, \tilde{Y}_i) = \begin{bmatrix} V & V \\ V^T & V + S_i \end{bmatrix} \tag{3.9}$$

$$0 <_L Var(\tilde{Y}_i, \tilde{P}) \tag{3.10}$$

X_i^1 is convex, unbounded, open, and clearly nonempty. We will show that the mapping from initial to updated beliefs takes this set into a convex, compact set, which we will use as the belief parameter set of the agents in the model. Let Q_i be any element of X_i; Q_i is a possible value for $Var(\tilde{F}, \tilde{Y}_i, \tilde{P})$. Let T_{1i} denote the mapping that takes Q_i to the parameters of the conditional distribution of $\tilde{F}|\tilde{Y}_i, \tilde{P}$. Specifically, $T_{1i} : X_i \to (\mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \times \mathbb{S}_+^N)$ maps Q_i to the matrices $(B_{0i}, B_{1i}, B_{2i}, V_i)$, which are uniquely determined by the conditional Gaussian distribution (Lemma 2.1):

$$\tilde{\mu}_i = E[\tilde{F}|\tilde{Y}_i, \tilde{P}] = \bar{F} + Cov(\tilde{F}, (\tilde{Y}_i, \tilde{P}))Var^{-1}(\tilde{Y}_i, \tilde{P}) \left[\begin{smallmatrix} \tilde{Y}_i - \bar{Y}_i \\ \tilde{P} - \bar{P} \end{smallmatrix} \right] \tag{3.11}$$

$$= B_{0i} + B_{1i}\tilde{Y}_i + B_{2i}\tilde{P} \tag{3.12}$$

$$V_i = Var(\tilde{F}|\tilde{Y}_i, \tilde{P}) = V - Cov(\tilde{F}, (\tilde{Y}_i, \tilde{P}))Var^{-1}(\tilde{Y}_i, \tilde{P})Cov((\tilde{Y}_i, \tilde{P}, \tilde{F})) \tag{3.13}$$

Constraint 3.10 guarantees that $Var(\tilde{Y}_i, \tilde{P})$ is nonsingular. By Corollary 2.1, this mapping is continuous. Plugging $\tilde{\mu}_i = B_{0i} + B_{1i}\tilde{Y}_i + B_{2i}\tilde{P}$ for $i = 1, ..., I$ into the equilibrium price condition 3.7 gives:

$$\tilde{P} = (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}(B_{0i} + B_{1i}\tilde{Y}_i + B_{2i}\tilde{P}) - \tilde{Z}) \tag{3.14}$$

$$= (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{0i} + \sum_i W_i^{-1}B_{1i}\tilde{Y}_i + \sum_i W_i^{-1}B_{2i}\tilde{P} - \tilde{Z}) \tag{3.15}$$

$$= (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{0i} + \sum_i W_i^{-1}B_{1i}\tilde{Y}_i - \tilde{Z}) + (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{2i})\tilde{P} \tag{3.16}$$

We can recursively expand $\tilde{P}$ on the right-hand side to get

$$\tilde{P} = (\sum_{k=0}^{\infty}\Theta^k)(\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{0i} + \sum_i W_i^{-1}B_{1i}\tilde{Y}_i - \tilde{Z}) \tag{3.17}$$

$$\Theta = (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{2i}) \tag{3.18}$$

We wish to show that the infinite sum $\sum_{k=0}^{\infty}\Theta^k$ converges; we can then write $\sum_{k=0}^{\infty}\Theta^k = (I-\Theta)^{-1}$. W_i and W_i^{-1} are bounded and bounded away from singular, and so is $(\sum_i W_i^{-1})^{-1}$. The coefficient of $\tilde{Z}$ is $-\sum_{k=0}^{\infty}\Theta^k(\sum_i W_i^{-1})^{-1}$, thus the variance contributed by the term containing $\tilde{Z}$ to $Var(\tilde{P})$ is

$$(\sum_{k=0}^{\infty}\Theta^k)(\sum_i W_i^{-1})^{-1}U(\sum_i W_i^{-1})^{-1}(\sum_{k=0}^{\infty}\Theta^k)^T \tag{3.19}$$

Since $\tilde{Z}$ is independent of the other random components of $\tilde{P}$ (the $\tilde{Y}_i$'s), $Var(\tilde{P})$ can be decomposed into a sum of the variance of the $\tilde{Z}$ term and the joint variance of the $\tilde{Y}_i$ terms. By Lemma 3.2 $Var(\tilde{P})$ is bounded; therefore the variance of the $\tilde{Z}$ term given in 3.19 must be bounded as well, and $\sum_{k=0}^{\infty}\Theta^k$ converges. This implies that $\max_i|\lambda_i(\Theta)| < 1$, and we can write $\sum_{k=0}^{\infty}\Theta^k = (I-\Theta)^{-1}$. Now we can compute the coefficients $(A_0, A_{11}, ..., A_{1I}, A_2)$ of the linear price equation 3.2:

$$A_0 = (I-\Theta)^{-1}(\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}B_{0i}) \tag{3.20}$$

$$A_{1i} = (I-\Theta)^{-1}(\sum_i W_i^{-1})^{-1}(W_i^{-1}B_{1i}) \tag{3.21}$$

$$A_2 = -(I-\Theta)^{-1}(\sum_i W_i^{-1})^{-1} \tag{3.22}$$

Finally, given $(A_0, A_{11}, ..., A_{1I}, A_2)$, we can compute the updated variance of $(\tilde{F}, \tilde{Y}_i, \tilde{P})$:

$$Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) = \begin{bmatrix} V & V & V(\sum_i A_{1i})^T \\ . & V+S_i & V(\sum_i A_{1i})^T + S_i A_{1i}^T \\ . & . & (\sum_i A_{1i})V(\sum_i A_{1i})^T + \sum_i A_{1i}S_iA_{1i}^T + A_2UA_2^T \end{bmatrix} \tag{3.23}$$

Let $T_2 : \prod_i(\mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \times \mathbb{S}_+^N) \to \prod_i \mathbb{S}_+^{3N}$ denote the mapping that takes the collection of

$(B_{0i}, B_{1i}, B_{2i}, V_i)$ for $i = 1, ..., I$ to a set of variance matrices $R_1, ...R_I$ using Eqn 3.23. Let T denote the mapping $T_2(\prod_i T_{1i})$.

We can now show that the variance bounds defining our desired belief parameter set exist, though it may be difficult to actually compute numerical bounds for a given set of parameter values.

Lemma 3.4 (variance bounds on private signal, price, and belief)**.** *For any* $(Q_1, ...Q_I)$*, where* $Q_i \in X_i^1$*, let* $(Q_1', ...Q_I') = T(Q_1, ...Q_I)$*. Then for each* Q_i'*, and using* $Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) = Q_i'$*: (i)* $Var(\tilde{P})$ *is bounded; (ii)* $Var(\tilde{Y}_i, \tilde{P})$ *is bounded away from singular; (iii)* $Var(\tilde{F}|\tilde{Y}_i, \tilde{P}) \geqslant_L \underline{V}$*.*

Therefore, given the parameters of the economy, there exist matrices $\underline{V}_{YP}^i \in \mathbb{S}_{++}^{2N}, \overline{V}_P \in \mathbb{S}_{++}^N$, such that $\underline{V}_{YP}^i$ is a lower bound for $Var(\tilde{Y}_i, \tilde{P})$, and $\overline{V}_P$ is an upper bound for $Var(\tilde{P})$. A nonsingular lower bound for $Var(\tilde{Y}_i, \tilde{P})$ is necessary to ensure that our mapping, which includes the matrix inverse, will not encounter a singular matrix. This implies that $(\tilde{Y}_i, \tilde{P})$ cannot be arbitrarily close to perfectly collinear, or equivalently, the conditional variance of $\tilde{P}|\tilde{Y}_i$ must be bounded away from singular.

3.1.2 The Set of Agents' Belief Variances

Let $X_i^2(\underline{V}_{YP}^i, \overline{V}_P)$ denote the set $\{Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) \in \mathbb{S}_+^{3N}\}$ subject to the following constraints:

$$Var(\tilde{F}, \tilde{Y}_i) = \begin{bmatrix} V & V \\ V^T & V + S_i \end{bmatrix} \tag{3.24}$$

$$0 \leqslant_L Var(\tilde{P}) \leqslant_L \overline{V}_P \tag{3.25}$$

$$\underline{V}_{YP}^i \leqslant_L Var(\tilde{Y}_i, \tilde{P}) \tag{3.26}$$

$$\underline{V} \leqslant_L Var(\tilde{F}|\tilde{Y}_i, \tilde{P}) \tag{3.27}$$

Lemma 3.5 (Convexity and compactness of agents' belief variance space)**.** *For any* $\underline{V}_{YP}^i \in \mathbb{S}_{++}^{2N}, \overline{V}_P \in \mathbb{S}_{++}^N$*,* $X_i(\underline{V}_{YP}^i, \overline{V}_P)$ *is closed, convex, and bounded.*

Constraint 3.25 ensures boundedness; the $0 \leqslant_L Var(\tilde{P})$ condition is simply positive semidefiniteness of $Var(\tilde{P})$. Constraint 3.26 ensures that $Var(\tilde{Y}_i, \tilde{P})$ is nonsingular, allowing us to take the matrix inverse when computing the distribution of $(\tilde{F}|\tilde{Y}_i, \tilde{P})$, while still defining a closed set.

Clearly, $X_i^2(\underline{V}_{YP}^i, \overline{V}_P) \subset X_i^1$. Therefore, T is a continuous mapping that takes $\prod_i X_i^2(\underline{V}_{YP}^i, \overline{V}_P)$

to itself. $\prod_i X_i^2(\underline{V}^i_{YP}, \overline{V}_P)$ is convex and compact, since it is the Cartesian product of convex and compact sets. It is nonempty since it contains at least the element $T(Q_1, ...Q_I)$ where each $Q_i \in X_i^1$ is

$$Q_i = \begin{bmatrix} V & V & 0 \\ . & V + S_i & 0 \\ . & . & S_i \end{bmatrix} \tag{3.28}$$

Since T is continuous, we can apply Brouwer's fixed point theorem, and a fixed point $Q^* = (Q_1^*, ...Q_I^*)$ exists, which must be a REE. Once we have Q^*, we can find the equilibrium price function by applying Eqns 3.12, 3.13, and 3.20 - 3.22.

4 Numerical Example and Multiple Equilibria

Here, we use Mann iteration to find an approximate fixed point of the mapping T. Since T is not known to be nonexpansive[3] (that is, a weakly contractive mapping) everywhere on our domain of interest, we cannot guarantee that this procedure converges, but in our tests it seems to work well for a variety of parameters. The source code for this algorithm is available upon request.

Suppose $\rho_1 = \rho_2 = 1$, $I = N = 2$, $S_1 = S_2 = I_2$, $\tilde{F} \sim N\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0.5\\ . & 1\end{smallmatrix}\right]\right)$, $\tilde{Z} \sim N\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0\\ . & 1\end{smallmatrix}\right]\right)$, and $\alpha_1 = \alpha_2 = 0.5$. We use the iterative procedure $x_{n+1} = (1 - \frac{1}{n})x_n + \frac{1}{n}T(x_n)$, and stop iterating when the Euclidean distance between successive iterates is less than 10^{-6}. By using random starting points, we have numerically found two equilibria (there may be more):

(i)

$$Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) = \begin{bmatrix} 1 & 0.5 & 1 & 0.5 & 0.597319 & 0.398735 \\ 0.5 & 1 & 0.5 & 1 & 0.398735 & 0.597319 \\ 1 & 0.5 & 2 & 0.5 & 0.86262 & 0.465452 \\ 0.5 & 1 & 0.5 & 2 & 0.465452 & 0.86262 \\ 0.597319 & 0.398735 & 0.86262 & 0.465452 & 0.680554 & 0.438109 \\ 0.398735 & 0.597319 & 0.465452 & 0.86262 & 0.438109 & 0.680554 \end{bmatrix} \tag{4.1}$$

[3] Note that a mapping T may satisfy the requirement for Brouwer's theorem that $T(X) \subset X$ for some set X, while still being expansive on a subset of X.

$$\tilde{\mu}_i = \begin{bmatrix} 0.295466 \\ 0.295466 \end{bmatrix} + \begin{bmatrix} 0.274761 & 0.0690959 \\ 0.0690959 & 0.274761 \end{bmatrix} \tilde{Y}_i + \begin{bmatrix} 0.482172 & 0 \\ 0 & 0.482172 \end{bmatrix} \tilde{P} \tag{4.2}$$

$$V_i = \begin{bmatrix} 0.402681 & 0.101265 \\ 0.101265 & 0.402681 \end{bmatrix} \tag{4.3}$$

$$\tilde{P} = \begin{bmatrix} 0.570587 \\ 0.570587 \end{bmatrix} + \begin{bmatrix} 0.265301 & 0.066717 \\ 0.066717 & 0.265301 \end{bmatrix} (\tilde{Y}_1 + \tilde{Y}_2) - \begin{bmatrix} 0.388817 & 0.0977786 \\ 0.0977786 & 0.388817 \end{bmatrix} \tilde{Z} \tag{4.4}$$

(ii)

$$Var(\tilde{F}, \tilde{Y}_i, \tilde{P}) = \begin{bmatrix} 1 & 0.5 & 1 & 0.5 & 0.0955325 & 0.336214 \\ 0.5 & 1 & 0.5 & 1 & -0.186054 & 0.374306 \\ 1 & 0.5 & 2 & 0.5 & 0.221239 & 0.435588 \\ 0.5 & 1 & 0.5 & 2 & -0.341934 & 0.511773 \\ 0.0955325 & -0.186054 & 0.221239 & -0.341934 & 0.272793 & 0.0118418 \\ 0.336214 & 0.374306 & 0.435588 & 0.511773 & 0.0118418 & 0.309886 \end{bmatrix} \tag{4.5}$$

$$\tilde{\mu}_i = \begin{bmatrix} 0.00948797 \\ -0.0103724 \end{bmatrix} + \begin{pmatrix} 0.374862 & 0.0517506 \\ 0.0938346 & 0.164442 \end{pmatrix} \tilde{Y}_i + \begin{bmatrix} 0.141774 & 0.220595 \\ -0.186571 & 0.318774 \end{bmatrix} \tilde{P} \tag{4.6}$$

$$V_i = \begin{bmatrix} 0.432879 & 0.102101 \\ 0.102101 & 0.369753 \end{bmatrix} \tag{4.7}$$

$$\tilde{P} = \begin{bmatrix} 0.0134933 \\ 0.00378235 \end{bmatrix} + \begin{bmatrix} 0.374862 & 0.0517506 \\ 0.0938346 & 0.164442 \end{bmatrix} (\tilde{Y}_1 + \tilde{Y}_2) - \begin{bmatrix} 0.090687 & 0.469108 \\ -0.587907 & 0.826878 \end{bmatrix} \tilde{Z} \tag{4.8}$$

Both equilibria are symmetric with respect to agents' beliefs. Equilibrium (i) is symmetric with respect to asset prices and all covariances are positive; a high price for asset 1 is correlated with a high return for asset 2, and vice versa. In contrast, equilibrium (ii) has a negative covariance between $\tilde{P}^1$ and $\tilde{F}^2$; a high price for asset 1 is correlated with a low return for asset 2. Equilibrium (ii) requires many more iterations to reach convergence compared to (i), indicating that it may be locally unstable.

5 Conclusion

Using a novel method, we have shown a linear REE exists in a finite-agent version of Admati (1985)'s multi-asset REE model. Our method is to show that the set of possible covariance matrices characterizing agents' beliefs is convex, compact and nonempty; we then show that the mapping from agents' initial beliefs to updated beliefs is continuous. By Brouwer's fixed point theorem, a fixed point (which must be a REE) exists. Although this method does not provide a closed-form solution, we still know that equilibrium beliefs are Gaussian and that Lemmas 3.3 and 3.1 hold in equilibrium. We have numerically shown that multiple equilibria may exist. This method can be applied to other rational expectations models with Gaussian uncertainty and where behavior is a linear function of agents' information.

This approach to existence of equilibria can be extended in two ways. First, in the static model, we can extend the type of uncertainty to other, non-Gaussian distributions. In particular, it is known that the exponential family of distributions has a parameter space that is an open, convex set. Second, while remaining within the CARA-Gaussian framework, another possibility is to extend this approach to models of strategic behavior with linear strategies, such as the model of Kyle (1989).

6 Appendix: Proofs

6.1 Preliminaries

Corollary 2.1 (Continuity of conditional Gaussian distribution).

Proof. The operations involved are: taking a submatrix of M; matrix addition and multiplication; and taking the matrix inverse. The matrix inverse is continuous over the set of nonsingular matrices, which is guaranteed by the constraint $M_{22} \in \mathbb{S}^n_{++}$; all the other operations are continuous over the set of all real-valued matrices. Therefore, the mappings are continuous. $\square$

Lemma 6.1 (Positive definiteness and Schur complement). *Suppose $M = \left[\begin{smallmatrix} A & C \\ C^T & B \end{smallmatrix}\right]$ is a symmetric, real-valued matrix. Then the following are equivalent: (i) $M >_L 0$; (ii) $B >_L 0$ and $A >_L CB^{-1}C^T$; (iii) $A >_L 0$ and $B >_L C^T A^{-1} C$. (Bernstein (2009), Prop 8.2.4)*

Lemma 6.2 (Covariance inequality). *Suppose $\tilde{x}, \tilde{y}$ are real-valued random variables with finite second moments. Then $|Cov(\tilde{x}, \tilde{x})|^2 \leqslant Var(\tilde{x})Var(\tilde{y})$ (Mukhopadhyay (2000), Thm 3.9.6).*

For $A \in \mathbb{S}^n_+$, let $\lambda_{min}(A), \lambda_{max}(A), d_{min}(A), d_{max}(A)$ denote the minimum and maximum eigenvalues and diagonal entries, respectively, of A.

Lemma 6.3 (Minimum and maximum eigenvalues of positive definite matrices). *Suppose $A, B \in \mathbb{S}^n_+$. Then:*

(a) $\lambda_{min}(A)I \leqslant_L A \leqslant_L \lambda_{max}I$. (Bernstein (2009), Corr. 8.4.2)

(b) $0 \leqslant \lambda_{min}(A) \leqslant d_{min}(A) \leqslant d_{max}(A) \leqslant \lambda_{max}(A)$ (Bernstein (2009), Corr. 8.4.7)

(c) $A \leqslant_L (<_L)\ B \Rightarrow \lambda_{min}(A) \leqslant (<)\ \lambda_{min}(B)$ and $\lambda_{max}(A) \leqslant (<)\ \lambda_{max}(B)$ (Bernstein (2009), Thm. 8.4.9)

(d) $A \leqslant_L B \Rightarrow \mathbf{tr}(A) \leqslant \mathbf{tr}(B)$ (Bernstein (2009), Corr. 8.4.10)

(e) The trace of A is equal to the sum of its eigenvalues. (Bernstein (2009), Fact 8.17.8)

Lemma 6.4 (Minimum and maximum eigenvalues of partitioned matrices). *Suppose $M \subset \mathbb{S}^{m+n}_{++} = Var(\tilde{X}, \tilde{Y})$ and is partitioned as in Lemma 2.1. Then:*

(a) $\lambda_{min}(M) \leqslant \lambda_{min}(Var(\tilde{X})) \leqslant \lambda_{max}(Var(\tilde{X})) \leqslant \lambda_{max}(M)$ and $\lambda_{min}(M) \leqslant \lambda_{min}(Var(\tilde{Y})) \leqslant \lambda_{max}(Var(\tilde{Y})) \leqslant \lambda_{max}(M)$. (Bernstein (2009), Corr. 8.4.6)

(b) $\lambda_{max}(M) \geqslant \lambda_{max}(Var(\tilde{X}|\tilde{Y}))$, $\lambda_{max}(M) \geqslant \lambda_{max}(Var(\tilde{Y}|\tilde{X}))$, $\lambda_{min}(M) \leqslant \lambda_{min}(Var(\tilde{X}|\tilde{Y}))$, $\lambda_{min}(M) \leqslant \lambda_{min}(Var(\tilde{Y}|\tilde{X}))$. (Zhang (2005), Corr. 2.3, and note that $Var(\tilde{X}|\tilde{Y}) = M/M_{22} = M_{11} - M_{12}M_{22}^{-1}M_{12}^T$).

Lemma 2.2 (Bounded positive semidefinite matrices).

Proof. (a) $\Rightarrow$ (b): By Lemma 6.3b, d and e, the eigenvalues of B are bounded, and therefore the eigenvalues and diagonal elements of A are bounded. By the covariance inequality 6.2, all off-diagonal entries of A are also bounded. (b) $\Rightarrow$ (c): Since $\mathbf{tr}(A)$ is bounded, the sum of A's eigenvalues are bounded and each is non-negative, therefore $\lambda_{max}(A)$ is bounded. (c) $\Rightarrow$ (a): Let $B = \lambda_{max}I$; then (a) is satisfied. $\square$

Lemma 2.3 (Positive definite matrices bounded away from singular).

Proof. (a) $\Rightarrow$ (b): By Lemma 6.3b, d and e, the eigenvalues of B are bounded away from zero, and therefore the eigenvalues of A are bounded away from zero. (b) $\Rightarrow$ (c): By Lemma 6.4b. (c) $\Rightarrow$ (a): By Lemma 6.5, $M_{11} = Var(\tilde{X}) \geqslant_L Var(\tilde{X}|\tilde{Y})$ and $M_{22} = Var(\tilde{Y}) \geqslant_L Var(\tilde{X}|\tilde{Y})$ are both bounded away from singular. Consider the given condition that $Var(\tilde{X}|\tilde{Y})$ is bounded away from singular. Then there exists $A \in \mathbb{S}^m_{++}$ such that

$$Var(\tilde{X}|\tilde{Y}) = M_{11} - M_{12}M_{22}^{-1}M_{12}^T \geqslant_L A >_L 0 \tag{6.1}$$

We want to show $\underline{M} \in \mathbb{S}^{m+n}_{++}$ exists such that $M \geqslant_L \underline{M} >_L 0$. Subtract $A/2$ from both sides of Eq. 6.1 to get $(M_{11} - \frac{A}{2}) - M_{12}M_{22}^{-1}M_{12}^T \geqslant_L \frac{A}{2} >_L 0$. Then

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \geqslant_L \begin{bmatrix} M_{11} - \frac{A}{2} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} >_L 0 \tag{6.2}$$

The first inequality holds because $\left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{smallmatrix}\right) - \left(\begin{smallmatrix} M_{11}-\frac{A}{2} & M_{12} \\ M_{12}^T & M_{22} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{smallmatrix}\right) \geqslant_L 0$. The second inequality holds by Lemma 6.1. The second matrix is the desired $\underline{M}$. $\square$

Corollary 2.2 (Inverse of bounded matrix is bounded away from singular).

Proof. The eigenvalues of A^{-1} are the inverse of the eigenvalues of A. $\lambda_{max}(A)$ is bounded iff $\lambda_{min}(A^{-1}) = \lambda_{max}(A)^{-1}$ is bounded away from singular. $\square$

Lemma 6.5 (Loewner ordering of conditional variance)**.** *Suppose $\tilde{X}, \tilde{Y}$ are jointly Gaussian random vectors. Then $Var(\tilde{X}|\tilde{Y}) \leqslant_L Var(\tilde{X})$.*

Proof. By the law of total variance:

$$Var(\tilde{X}) = E_{\tilde{Y}}[Var(\tilde{X}|\tilde{Y})] + Var_{\tilde{Y}}\left(E[\tilde{X}|\tilde{Y}]\right) \tag{6.3}$$

For a jointly Gaussian distribution, $Var(\tilde{X}|\tilde{Y})$ is deterministic, so $E_{\tilde{Y}}[Var(\tilde{X}|\tilde{Y})] = Var(\tilde{X}|\tilde{Y})$. $Var(\tilde{X}) - Var(\tilde{X}|\tilde{Y}) = Var_{\tilde{Y}}\left(E[\tilde{X}|\tilde{Y}]\right)$ which is positive definite, therefore $Var(\tilde{X}|\tilde{Y}) \leqslant_L Var(\tilde{X})$. $\square$

Lemma 6.6 (Loewner lower bound of a closed set)**.** *Suppose X is a closed subset of $\mathbb{S}^n_{++}$. Then there exists $\underline{M} \in \mathbb{S}^n_{++}$ such that for each $M \in X, M \geqslant_L \underline{M} >_L 0$.*

Proof. Let $\underline{\lambda} = \min_{M \in X} \lambda_{min}(M)$, the smallest eigenvalue across all $M \in X$; then $\underline{\lambda}$ is bounded away from singular. By Lemma 6.3a, $M \geqslant_L \underline{\lambda} I >_L 0$ for all $M \in X$. □

Lemma 6.7 (concavity of conditional variance)**.** *Suppose $M \subset \mathbb{S}^{m+n}_{++} = Var(\tilde{X}, \tilde{Y})$ and is partitioned as in Lemma 2.1. Then:*

(a) the map $M \to Var(\tilde{X}|\tilde{Y}) = M_{11} - M_{12}M_{22}^{-1}M_{12}^T$ is concave.

(b) The upper level set of this map with respect to a fixed $A \in \mathbb{S}^m_+$, $G(A) = \{M \in \mathbb{S}^{m+n}_+ | M_{22} >_L 0, Var(\tilde{X}|\tilde{Y}) \geqslant_L A\}$, is a convex, open set.

(c) Imposing the additional constraint $M_{22} \geqslant_L B$ for some fixed $B \in \mathbb{S}^n_{++}$ results in $H(A, B) = \{M \in \mathbb{S}^{m+n}_+ | M_{22} \geqslant_L B >_L 0, Var(\tilde{X}|\tilde{Y}) \geqslant_L A\}$, which is a convex, closed set.

Proof. For (a), see Corollary 1.5.3 in Bhatia (2007). For (b), the upper level set of a concave function is a convex set; it is open since the constraint $M_{22} >_L 0$ defines an open set. For (c), we replace the previous constraint with one that defines a closed set. □

6.2 Proofs

Theorem 2.1 (Convexity of set of covariance matrices subject to constraints).

Proof. We proceed by induction. Let V_x^0 denote the set of possible $Var(\tilde{X})$ before any constraints have been added, and let V_x^i denote the set of valid covariance matrices after constraints $1, ..., i$ have been added; we assume V_x^i is convex and closed. We form V_x^{i+1} by defining an additional constraint and taking the intersection of the set it defines with V_x^i.

(a) Suppose we add a type (a) constraint: for fixed $A, B \in \mathbb{S}^n_+$, we impose $A \leqslant_L Var(\tilde{X}_I) \leqslant_L B$. Let C denote the set $\{Var(\tilde{X}_I) \in \mathbb{S}^I_+ | A \leqslant_L Var(\tilde{X}_I) \leqslant_L B\}$, which is clearly a closed, bounded, and convex subset of $\mathbb{S}^n_+$. Let $V_x^{i+1} = V_x^i \cap C$; this intersection is closed and convex.

(b) Suppose we add a type (b) constraint: (i) $Var(\tilde{X}_J) \geqslant_L A >_L 0$; (ii) $Var(\tilde{X}_I|\tilde{X}_J) \geqslant_L B \geqslant_L 0$. Let C denote the set $\{Var(\tilde{X},\tilde{Y})|Var(\tilde{X}_J) \geqslant_L A >_L 0, Var(\tilde{X}_I|\tilde{X}_J) \geqslant_L B \geqslant_L 0\}$. By Lemma 6.7, C is convex and closed, and so is $V_x^{i+1} = V_x^i \cap C$.

$V_x^0 = \mathbb{S}_+^n$, which is convex and closed. By induction, the property holds for any i.

Furthermore, suppose that after imposing constraints $1, ...i$, each of $Var(\tilde{x}_1), ...Var(\tilde{x}_n)$ is bounded. By the covariance inequality (Lemma 6.2), every off-diagonal element must also be bounded, so V_x^i is bounded. $\square$

Lemma 3.3 (Expectation of beliefs and prices).

Proof. First result: apply the law of iterated expectations to $E[E(\tilde{F}|\tilde{Y}_i,\tilde{P})]$. Second result: take expectations of Eqn 3.7 and plug in $\bar{F}$ for each $E[\tilde{\mu}_i]$ and $\bar{Z}$ for $E[\tilde{Z}]$:

$$E[\tilde{P}] = (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}E[\tilde{\mu}_i]) - (\sum_i W_i^{-1})^{-1}E[\tilde{Z}]] \tag{6.4}$$

$$= (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1}\bar{F}) - (\sum_i W_i^{-1})^{-1}\bar{Z} \tag{6.5}$$

$$= (\sum_i W_i^{-1})^{-1}(\sum_i W_i^{-1})\bar{F} - (\sum_i W_i^{-1})^{-1}\bar{Z} = \bar{F} - (\sum_i W_i^{-1})^{-1}\bar{Z} \tag{6.6}$$

$\square$

Lemma 3.1 (Boundedness of agent's belief variance).

Proof. Apply Lemma 6.5 twice. First, condition $(\tilde{F}|\tilde{Y}_i,\tilde{P})$ on $(\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$ to get $Var(\tilde{F}|\tilde{Y}_i,\tilde{P}) \geqslant_L Var(\tilde{F}|\tilde{P},\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$. Then $(\tilde{F}|\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$ is independent of $(\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$; since $\tilde{P}$ is a linear function of $(\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$, then $(\tilde{F}|\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z})$ is also independent of $\tilde{P}$. It makes no difference whether $\tilde{P}$ is conditioned on or not, and $V_i \geqslant_L Var(\tilde{F}|\tilde{P},\tilde{Y}_1,...,\tilde{Y}_I,\tilde{Z}) = \underline{V}$. For the second inequality, condition $\tilde{F}$ on $(\tilde{Y}_i,\tilde{P})$ to get $Var(\tilde{F}|\tilde{Y}_i,\tilde{P}) \leqslant_L Var(\tilde{F})$. $\square$

Lemma 3.2 (Variance of price is bounded).

Proof. By the law of total variance:

$$Var(\tilde{F}) = E_{\tilde{Y}_i,\tilde{P}}[Var(\tilde{F}|\tilde{Y}_i,\tilde{P})] + Var_{\tilde{Y}_i,\tilde{P}}(E[\tilde{F}|\tilde{Y}_i,\tilde{P}]) \tag{6.7}$$

$$V = V_i + Var(\tilde{\mu}_i) \tag{6.8}$$

Therefore, $Var(\tilde{\mu}_i)$ is bounded. By Lemma 3.1, each V_i and V_i^{-1} is bounded $\Rightarrow$ each W_i and W_i^{-1} is bounded $\Rightarrow$ the coefficients of $\tilde{\mu}_1, ..., \tilde{\mu}_I, \tilde{Z}$ in Eqn 3.7 are bounded. Since $\tilde{P}$ is a linear combination of random variables with bounded variance and bounded coefficients, $Var(\tilde{P})$ must be bounded. □

Lemma 3.4 (Variance bounds on private signal, price, and belief).

Proof. (i) and (iii) follow from the fact that the mapping T involves computing the conditional distribution of $(\tilde{F}|\tilde{Y}_i,\tilde{P})$ in Eqns. 3.12 and 3.12, and applying Lemmas 3.1 and 3.2. (ii): Consider the matrix Θ given by Eqn. 3.18; for any eigenvalue $\lambda_i(\Theta)$, we have $|\lambda_i(\Theta)| < 1$ (see discussion following Eqn. 3.19). Then the corresponding eigenvalue of $(I-\Theta)^{-1}$ is $\frac{1}{1-\lambda_i(\Theta)}$, which is bounded away from zero. Therefore, the variance of $A_2\tilde{Z}$, where $A_2 = -(I-\Theta)^{-1}(\sum_i W_i^{-1})^{-1}$ (Eqn. 3.22) is bounded away from singular. Since $Var(A_2\tilde{Z}) = Var(\tilde{P}|\tilde{Y}_1, ...\tilde{Y}_I)$, and by Lemma 6.5, $Var(\tilde{P}|\tilde{Y}_i) \geqslant_L Var(\tilde{P}|\tilde{Y}_1, ...\tilde{Y}_I)$, then $Var(\tilde{P}|\tilde{Y}_i)$ is bounded away from singular. Lemma 2.3 completes the result. □

Lemma 3.5 (Convexity and compactness of agents' belief variance space).

Proof. We apply Theorem 2.1. Constraint 3.24 is a type (a) constraint equivalent to

$$\begin{bmatrix} V & V \\ V^T & V + S_i \end{bmatrix} \leqslant_L Var(\tilde{F},\tilde{Y}_i) \leqslant_L \begin{bmatrix} V & V \\ V^T & V + S_i \end{bmatrix} \tag{6.9}$$

Constraint 3.26 is a type (a) constraint. Constraints 3.25 and 3.27 are a type (b) constraint. Thus, the result holds. □

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