Maturity Mismatch and Fractional-Reserve Banking

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Abstract

In this paper, we present a dynamic model of a banking firm based on inventory management of stochastic cash flows. The bank takes deposits and makes loans, which may have different maturities. The model provides a parsimonious explanation of dividend payouts, maturity mismatch, quantities of credit, and bankruptcy; mismatch arises non-strategically, from profit maximization. Numerical experiments show that maturity mismatch can be optimal if the level of uncertainty in withdrawals is not too high. Uncertainty of both loan repayments and withdrawals can result in a bank that optimally fails in finite time.

Keywords: maturity mismatch, fractional reserve, inventory management, banking firm

1 Introduction

The financial crisis of 2007-2009 demonstrated that financial intermediaries play a critical, if not yet well-understood, role in the economy. However, our theoretical understanding of how intermediaries actually work is currently incomplete. This paper contributes to the theoretical literature by developing a model of a banking firm that is fundamentally based on cash flows. From the bank's point of view, a loan is a cash outflow today, followed by a promise of a cash inflow at some point in the future; a deposit is the reverse. The bank solves a stochastic, dynamic inventory management problem, maximizing the profit stream that can be generated from deposits and loans. This modeling approach has a number of benefits: it is dynamic; bank operations in the model have interpretations that match everyday intuition about borrowing and lending; and maturity mismatch, which may be a puzzle to explain in other models, emerges naturally from optimizing behavior. We solve the model numerically and use simulation to explore how behavior changes in response to changes in parameters. We find that maturity mismatch can be optimal if the level of uncertainty in withdrawals is

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not too high. By itself, uncertainty in withdrawals does not necessarily cause an optimally behaving bank to fail, but it can when paired with uncertainty in repayments.

1.1 Modeling the banking firm

A seemingly simple (yet unresolved) question is how to model the banking firm itself. How does a bank make money? What are its choice variables and objective function? One modeling approach is to treat a bank like an industrial firm, with inputs, outputs, and a production function. In the Monti-Klein model (Klein (1971), Monti (1972)), inputs are the amount of deposits and loans, which "produces" a management cost according to some cost function. This approach has been extended by adding variables thought to be relevant to the inputs or outputs; for example, Hughes, Mester, and Moon (2001) add equity capital and labor as inputs. In macroeconomics, this approach is convenient since it allows modelers to include a financial sector in the same way that firms are typically modeled. Some recent examples from the macroeconomics literature are Goodfriend and McCallum (2007), in which the inputs are labor (used for monitoring loans) and collateral, while the output is the amount of loans; and Christiano, Motto, and Rostagno (2010), where inputs are labor, "capital services", and excess reserves, which produce "transaction services".

A basic conceptual difficulty with this approach is that it is not obvious what inputs and outputs are for a financial intermediary, or how one is transformed into the other. Are demand deposits inputs, outputs, or something else? How can we quantify the services that a bank's customers buy, and at what price are they sold? In addition, these models do not capture the possibility of bankruptcy, which is a basic concern of any financial institution.

Our approach is to model a bank as a dynamic, stochastic inventory management problem; cash inflows are generated by new deposits and loan repayments, while cash outflows are generated by deposits being withdrawn and new lending. Profits are a cash outflow that is extracted from the current inventory of cash reserves; bankruptcy occurs when the firm is unable to satisfy its cash outflow obligations (i.e. when its cash reserves fall below zero). In this way, we explicitly detail the mechanism by which one type of asset is transformed into another.

1.2 Maturity mismatch, fractional-reserve banking, and liquidity risk

The business of traditional banking is usually described as follows:

- A bank borrows from depositors, and lends to borrowers.
- It lends at a higher interest rate at which it borrows; this interest rate spread is the source of the bank's profits.
- However, loans have a longer duration than deposits (which can be withdrawn at any time).

The process of "borrowing short and lending long" is known as maturity transformation or mismatch; a bank that engages in this practice will not hold enough cash on hand to satisfy all its liabilities, which gives rise to the possibility of bank runs and panics. This system of fractional-reserve banking and its attendant risks is one of the oldest issues in economics. Why do banks engage in this risky process, which can result in the firm's extinction? Why not match the maturity of debts, or use equity capital to finance loans, either of which would eliminate the risk of bankruptcy? In an interview, Raghuram Rajan summarizes the problem:

> "Why the private banking sector has never chosen safe narrow banking (with finance companies issuing long-term liabilities and making illiquid loans) is really the puzzle of the ages." (Feldman 2009)

We propose that an inventory management perspective can help resolve this puzzle. A basic result in inventory theory is that when demand is random, it is optimal to hold less stock than is necessary to satisfy maximum possible demand. In our model, maturity mismatch is a dynamic outcome of optimally balancing inflows and outflows from recurring deposits and loans.

2 Related Literature

The literature on banking is vast. Freixas and Rochet (2008) and Gorton and Winton (2003) provide recent surveys of the banking literature, mostly pre-financial crisis. In this section, we show how this paper relates to the existing literatures on inventory management, insurance, and maturity mismatch.

2.1 Banking and Inventory Management

The idea of banking as inventory management goes back to Edgeworth (1888), which posed the question of how much reserves a bank should hold to cover stochastic withdrawals in a one-period setting. This problem, now known as the "newsvendor problem" (Porteus 2002), more generally applies to any situation where the level of stock must be chosen, subject to random demand. It generally turns out to be optimal to hold less stock than necessary to satisfy the maximum possible demand; equivalently, it is optimal to accept that underage (running out of stock) may occur with positive probability. For example, airlines find it optimal to overbook (i.e. promise more seats than they actually have).

We note two banking models that are similar in essence. Ho and Saunders (1981) model a bank as a market-maker facing random arrivals of borrowers and depositors. Prisman, Slovin, and Sushka (1986) introduce liquidity shocks into the Monti-Klein model. Both of these papers are static; the natural next step would be to extend these models into a dynamic setting. Before the financial crisis, there were few dynamic models of banking firms; O'Hara (1983) and Buchinsky and Yosha (1995) have dynamic models similar in spirit to the one in this paper, although they do not have maturity mismatch, and we explicitly find the solution to the bank's dynamic programming problem numerically. Milne (2004) models bank capital as an inventory process; our model treats all aspects of the bank (reserves, assets, and liabilities) as part of the inventory process.

2.2 "Optimal Dividends"

This paper also draws upon the literature on the "optimal dividends" problem of insurance. In this problem, an insurance firm with an initial cash reserve faces a stream of stochastic cash inflows (e.g. premium payments) and outflows (e.g. claims payouts); *ruin* occurs when its cash reserve falls below zero. The firm's objective is to maximize the expected discounted dividend payout until ruin. This problem was posed in insurance by de Finetti (1957) and in economics by Shubik and Thompson (1959) [1]. Dutta and Radner (1999) use a continuous-time version to investigate the tradeoff between profit maximization and bankruptcy risk. We find it useful to adapt this model because it is dynamic and incorporates intuitive notions of profitability and bankruptcy, as well as the inventory management aspect we wish to explore. The original problem specifies only the net earnings in a period; we will decompose this into separate cash inflows and outflows from banking activity. This model can also be viewed as an asset-liability management problem in which the transformation between cash and assets/liabilities is explicitly modeled.

2.3 Maturity Mismatch and Transformation

We contrast this paper with some leading models of maturity mismatch. Diamond and Dybvig (1983) interprets a bank as a provider of liquidity insurance, but do not model the bank itself as a self-interested agent; we model the bank as a profit-maximizing firm, and show how mismatch can arise from optimal behavior. In Calomiris and Kahn (1991) and Diamond and Rajan (2001), short-term debt is a bad that banks strategically take on as a signaling or self-disciplining device; in our model, the level of mismatch is non-strategically determined by a tradeoff between profits and bankruptcy risk.

The financial crisis has stimulated new research on maturity mismatch. Farhi and Tirole (2012) show how a policy of monetary intervention can cause collective moral hazard, leading to excessive maturity mismatch. Morris and Shin (2010) develop a global games model to examine liquidity risk. In He and Xiong (2012), a bank with a long-lived asset is financed with many short-term debts of random maturity; when each short-term debt matures, the creditor has the choice of whether to roll over the debt or withdraw its money. They show that a coordination problem arises among the bank's creditors, which can lead to liquidation of otherwise solvent firms. Brunnermeier and Oehmke (2012) model a financial institution that finances a long-term project with short or long-term debt; a "rat race" can arise in which a financial institution has the incentive to switch creditors from long to short-term debt. Most of these papers model mismatch as a game between creditors and perhaps the bank itself. In contrast, we focus on how mismatch can arise even without strategic behavior; rather, it is central to the firm's business model. An additional difference is that the previous papers focus on a single project, while we model the bank's choice to invest in future projects.

Another name for maturity mismatch is maturity transformation. A bank that takes short-term deposits and makes long-term loans has effectively consumed some of the supply of short-term lending, and produced additional supply of long-term lending in the economy. The degree

[1] Recent surveys of the optimal dividends problem in discrete and continuous time, respectively, are Avanzi (2009) and Albrecher and Thonhauser (2009).

to which maturity transformation takes place can affect the effectiveness of monetary policy transmission via the yield curve, as well as the amount of long-term investment undertaken in the economy. Although maturity transformation is an old idea, there have been few attempts to model the mechanism by which one duration is transformed into another; one recent paper that does is Vayanos and Vila (2009), which presents a "preferred habitat" model of the term structure of interest rates. In that paper, arbitrageurs choose a portfolio of bonds to maximize mean-value preferences over market value; they do not model bankruptcy. Our paper models maturity transformation without a securities market, and allows for the possibility of bankruptcy.

2.4 Application to Other Financial Activity

Finally, while banks are of obvious interest, the inventory management approach can be used to model a wide variety of problems that involve recurring, predictable inflows and outflows. Many financial activities can be thought of as inventory problems, such as:

- *Insurance*: The investor and insurance executive Warren Buffett describes the insurance business as follows (Buffett 2010):

  > "Insurers receive premiums upfront and pay claims later. In extreme cases, such as those arising from certain workers' compensation accidents, payments can stretch over decades. This collect-now, pay-later model leaves us holding large sums - money we call "float" - that will eventually go to others. Meanwhile, we get to invest this float for Berkshire's benefit."

- *Rehypothecation*: Prime brokers make loans to clients (e.g. hedge funds) secured by collateral; the practice of re-pledging the collateral by the prime broker for its own borrowing is called rehypothecation (Duffie 2010). Rehypothecation is a source of income for prime brokers, but can lead to the risk of runs by its clients.

- *Ponzi schemes*: The classic definition of a Ponzi scheme is an operation in which old investors are paid off by funds from new investors.

In each of these cases, the intermediary gains temporary possession of something that must be returned, but is able to make use of it (and possibly lose it) in the meantime. A shortcoming of this approach is that analytic solutions are known only for some simple cases; we solve our model numerically and attempt to build intuition by comparing optimal behavior over a range of parameter values.

The rest of this paper is organized as follows. Section 3 describes the basic "optimal dividends" problem, and develops a model of a banking firm based on that problem. Section 4 presents two numerical examples to illustrate optimal behavior. Section 5 presents comparative statics through simulation of the model. Section 6 discusses the results. Section 7 concludes.

3 The Model

3.1 The "Optimal Dividends" Problem of Insurance

To illustrate the ideas behind the model, we consider the "optimal dividends" problem of insurance. Time is discrete and infinite. At the beginning of period t, a firm has cash reserve M_t and will experience a net cash inflow of ΔY_t, where $\Delta Y_t = 1$ with probability p and -1 with probability $1-p$. The firm can choose to extract an amount $X_t \leq M_t$ from the current cash reserve; this is the firm's profit and is paid out to the firm's owners as dividends. The firm's owners gain per-period utility, assumed to be linear: $u(X_t) = X_t$. After cash flows are realized, the next period's beginning cash reserve is calculated: $M_{t+1} = M_t - X_t + \Delta Y_t$. If $M_{t+1} < 0$, the firm is bankrupt (*ruined* in the insurance literature), ceases operations, and pays out zero in all future periods; otherwise, it continues into the next period. The *time of ruin* is then $\tau = \inf\{t : M_t < 0\}$. The firm's objective is to maximize the expected sum of discounted dividend payments (which is also the valuation of the firm under the Gordon model):

$$V(M_0) = \max_{X_t} E_0 \left[\sum_{t=0}^{\tau-1} \beta^t X_t \right]$$

The Bellman equation for this problem is:

$$V(M_t) = \max_{X_t \in [0, M_t]} E\left[X_t + \beta E V(M_{t+1})\right]$$

subject to

$$V(M < 0) = 0$$

$$M_{t+1} = M_t - X_t + \Delta Y_t$$

This problem was introduced in de Finetti (1957) and Shubik and Thompson (1959); it turns out that the optimal strategy for this specific case is a *barrier strategy*, in which the optimal payout is zero if M_t is below some threshold level M^*, where M^* is determined by the probability distribution of ΔY_t; otherwise, the optimal payout is $M_t - M^*$, which immediately brings M_t down to the threshold level. Under the optimal strategy, ruin occurs in finite time almost surely. It is possible for M^* to be zero, in which case the optimal strategy is to immediately "loot" the firm of all its cash. In the general case, the optimal strategy is a *band strategy* (Porteus 1977).

We can extend this approach to model a bank which takes deposits and makes loans. Banks have two stochastic inflows (new deposits and loan repayments) and two stochastic outflows (withdrawals and new loans) with known distributions. The bank's choices may affect the distribution of cash flows.

This model bears some similarities to dynamic models used in corporate finance and asset pricing, with an important difference. In those models, a firm goes bankrupt when its equity value or net market value falls below zero; here, bankruptcy occurs when cash reserves falls below zero. We will not deal with asset markets or market valuation in this model; we assume that loans must be held to maturity and there are no secondary markets.

3.2 The Bank

We take the same basic approach to model a bank; however, we model inflows and outflows separately. This requires keeping track of the amount of promises that have been made in the past as state variables. We assume loans and deposits are variable-rate and compounded each period, and duration is a random variable. We state the dynamic programming problem:

Time is discrete and infinite. The bank begins each period with the following state variables:

- *Cash reserve*: M_t
- *Loans made*: Let L_t be the total face value of loans made; i.e. if all loans were to be repaid this period, L_t would be the amount repaid.
- *Deposit liabilities*: Let D_t be the total face value of deposits. If all deposits were to be withdrawn this period, D_t would be the amount the bank has to pay.
- *Population*: P_t

Note that L_t and D_t are not market values, but a stock of promised cash flows. We assume that the amount of loans and deposits that mature, as well as the amount of new loans and deposits, are proportional to their current values. L_t and D_t will undergo (possibly stochastic) geometric growth each period; together with linear utility, this implies that the value function will be homogenous of degree one. New deposits come from a population with exogenous growth rate g_t^p. In each period, the sequence of events (see Figure 1) is as follows:

1. The bank chooses the dividend payout $X_t \in [0, M_t]$, and the amount of new loans made, $\gamma_t^L L_t$, where $\gamma_t^L \geq 0$.
2. The bank gains per-period utility $u(X_t) = X_t$.
3. Cash inflows and outflows are realized. α^L and γ^D are exogenous constants; $\boldsymbol{\alpha_t^D}$ is a random variable.
    - *New loans made*: $\gamma_t^L L_t$
    - *Loans repaid*: $\alpha^L L_t$
    - *New deposits*: $\gamma^D P_t$
    - *Withdrawals*: $\boldsymbol{\alpha_t^D} D_t$, where $\boldsymbol{\alpha_t^D}$ is random and $P(\boldsymbol{\alpha_t^D} > \alpha^L) = 1$.
4. Next period's cash reserve, M_{t+1}, is determined by adding all cash inflows and outflows:

$$M_{t+1} = M_t - X_t + L_t(\alpha^L - \gamma_t^L) - D_t(\boldsymbol{\alpha_t^D} - \gamma^D \frac{P_t}{D_t}) \tag{1}$$

    If $M_{t+1} \leq 0$, then the bank is bankrupt, ceases operating, and receives a zero payout in all future periods.

5. Interest is compounded and state variables are updated. The interest rates on loans and deposits, r^L and r^D, are exogenous constants.

$$L_{t+1} = (1 - \alpha^L + \gamma_t^L)(1 + r^L)L_t \tag{2}$$

$$D_{t+1} = (1 - \boldsymbol{\alpha_t^D} + \gamma_t^D \frac{P_t}{D_t})(1 + r^D)D_t \tag{3}$$

$$P_{t+1} = g_t^p P_t \tag{4}$$

6. The next period begins.

Maturity mismatch is captured by the parameters α^L and $\boldsymbol{\alpha_t^D}$. To motivate this, we can imagine a continuum of small loans and deposits; each one has an independent probability of being repaid or withdrawn in a given period. Loan duration is then geometrically distributed with an expected duration of $1/\alpha^L$; by the law of large numbers, the fraction of all loans that are repaid in a period is α^L. We would like deposits to have a shorter average duration, while still having the total amount of withdrawals be uncertain. We do this by having $\boldsymbol{\alpha_t^D}$ be random, but always being greater than α^L.

The bank's problem is to maximize the expected sum of discounted payouts until the time of bankruptcy $\tau = \inf\{t : M_t \leq 0\}$:

$$V(M_0, L_0, P_0, D_0) = \max_{X_t, \gamma_t^L} E_0 \left[\sum_{t=0}^{\tau-1} \beta^t X_t \right]$$

The Bellman equation for this problem is:

$$V(M_t, L_t, P_t, D_t) = \max_{X_t, \gamma_t^L} [X_t + \beta E V(M_{t+1}, L_{t+1}, P_{t+1}, D_{t+1})]$$

subject to the transition equations 1-4 and the boundary condition $V(M_t \leq 0, \cdot, \cdot, \cdot) = 0$.

3.3 First-order conditions

Rewrite the Bellman equation as

$$V(M_t, L_t, P_t, D_t) = \max_{X_t, \gamma_t^L} [X_t + \beta Pr(M_{t+1} > 0) V(M_{t+1}, L_{t+1}, P_{t+1}, D_{t+1} | M_{t+1} > 0)] \tag{5}$$

where

$$\begin{aligned} Pr(M_{t+1} > 0) &= Pr(M_t - X_t + L_t(\alpha^L - \gamma_t^L) - D_t(\boldsymbol{\alpha_t^D} - \gamma^D \frac{P_t}{D_t}) > 0) \\ &= Pr(\boldsymbol{\alpha_t^D} < \frac{1}{D_t} \left[M_t - X_t + L_t(\alpha^L - \gamma_t^L) \right] + \gamma^D \frac{P_t}{D_t} \end{aligned} \tag{6}$$

The probability in equation 6 is a function of the control variables; denote it as $P(X_t, \gamma_t^L)$. Assume for the moment that it is continuous. The first-order condition for the right-hand side of the Bellman equation 5 with respect to X_t is

$$\frac{\partial}{\partial X_t} = 0 = 1 + \beta \left[\frac{\partial P}{\partial X_t}(X_t, \gamma_t^L) V(0_+, L_{t+1}, P_{t+1}, D_{t+1}) - P(X_t, \gamma_t^L) \frac{\partial V}{\partial M_{t+1}}(M_{t+1}, L_{t+1}, P_{t+1}, D_{t+1}) \right]$$

The second term on the right hand side is the usual marginal benefit to saving, multiplied by the probability of survival. The first term on the right hand side is the marginal probability of survival, multiplied by the value at the boundary of bankruptcy; it reflects the fact that a marginal increase in payouts will also marginally increase the probability of going bankrupt, decreasing future expected value. This term makes it difficult to find an analytical solution to the problem, since the first-order condition depends on the form of the probability distribution and the value function at the boundary. Although solutions to the "optimal dividends" problem are known for some special probability distributions (Avanzi (2009) and Albrecher and Thonhauser (2009) review the known solutions), we do not know if they extend to our problem formulation, where the distribution can depend on the state variables and there is more than one control variable. We therefore resort to numerical methods.

3.4 Elimination of a state variable

We show that $V(M_t, L_t, P_t, D_t)$ is homogenous of degree one. Refer back to the probability in equation 6; this probability is unchanged if M_t, L_t, P_t, D_t and X_t are divided by a constant. The transition equations 1-4, likewise still hold if $M_{t+1}, L_{t+1}, P_{t+1}$, and D_{t+1} are divided by the same constant. Assume that $V(\cdot)$ in the right-hand side of 5 is homogenous of degree one; it is then true for the left-hand side as well.

We can then eliminate a state variable by dividing by D_t. Let $x_t = \frac{X_t}{D_t}, m_t = \frac{M_t}{D_t}, l_t = \frac{L_t}{D_t}$, and $p_t = \frac{P_t}{D_t}$. These quantities can be thought of as amounts per dollar of liabilities. m_t is the ratio of cash to liabilities; it can also be thought of as the fraction in fractional-reserve banking. l_t is the ratio of assets to liabilities, and v is the ratio of firm value to liabilities. Let $\boldsymbol{g_t^D} = \frac{D_{t+1}}{D_t} = (1 - \boldsymbol{\alpha_t^D} + \gamma^D \frac{P_t}{D_t})(1 + r^D)$, the growth factor of deposits. We can restate the problem with the Bellman equation

$$v(m_t, l_t, p_t) = \max_{x_t \in [0, m_t], \gamma_t^L} \left\{ x_t + \beta E \left[\boldsymbol{g_t^D} v(m_{t+1}, l_{t+1}, p_{t+1}) \right] \right\} \tag{7}$$

subject to boundary condition $v(m \leq 0, \cdot, \cdot) = 0$ and transition equations

$$m_{t+1} = \frac{1}{g_t^D}(m_t - x_t + l_t(\alpha^L - \gamma_t^L) - (\boldsymbol{\alpha_t^D} - \gamma^D p_t)) \tag{8}$$

$$l_{t+1} = \frac{1}{g_t^D}(1 + \gamma_t^L - \alpha^L)(1 + r^L)l_t \tag{9}$$

$$p_{t+1} = \frac{g_t^p}{g_t^D} p_t \tag{10}$$

where $v(m_t, l_t, p_t) = v(\frac{M_t}{D_t}, \frac{L_t}{D_t}, \frac{P_t}{D_t}) = D_t V(\frac{M_t}{D_t}, \frac{L_t}{D_t}, \frac{P_t}{D_t}, 1)$. v may not exist for certain parameter values; for example, if $1/\beta$ is low compared to the interest that can be earned from lending, the bank will save everything and never pay out a dividend. We assume that $g_t^P = 1$ and $\boldsymbol{\alpha_t^D}$ has two outcomes, α_{low}^D and α_{high}^D, with respective probability $1 - q$ and q.

If we can find the solution to this problem, we will have a model that predicts optimal payout, lending, and cash buffering behavior in a given period; by simulating shocks from the environment, we can also predict the bank's optimal dynamic behavior over time. In particular, we can simulate the evolution over time of the following variables:

- m_t, which can be interpreted as the fraction of liabilities held as liquid assets (i.e. the fractional reserve). A lower m_t implies the bank operates with less of a cash buffer.

- l_t, which can be interpreted as the ratio of assets to liabilities. A higher l_t implies the bank operates with a larger balance sheet.

- γ_t^L, the choice of lending as a fraction of beginning loans. A higher γ_t^L implies higher lending activity and credit supply; if $\gamma_t^L = 0$ for any possible initial state, then it is not optimal to lend at all, and the model thus predicts no banking activity given this particular environment.

- v, firm value per dollar of liabilities. A higher v implies that a particular environment, defined by the parameters, makes banking activity more or less valuable.

- Time until bankruptcy. A longer observed time until bankruptcy implies that bank failures should be correspondingly rarer; if a bank's lifetime is effectively unlimited, then this would predict that bank failures do not occur, given the particular environment.

These results can provide theoretical insight into the question of maturity mismatch. If, given an environment where loans have a longer maturity than deposits, we observe zero lending; then it is not optimal to lend, and mismatch will not occur. On the other hand, if we observe positive lending, then it is optimal for the bank to engage in mismatch. Observing dynamic behavior can also tell us about the safety of such a practice. It may be that a bank engages in mismatch but lives forever; in that case, mismatch can theoretically be performed safely, and in fact it is optimal to do so. On the other hand, we may see a bank engage in mismatch, but risk bankruptcy in finite time. In that case, profit-maximizing banks, if left to themselves, will periodically fail; if this is socially inefficient, then regulation may be necessary.

4 Numerical Solution and Illustration

The dynamic problem described by equations 7 - 10 has three continuous state variables: the cash/liabilities ratio m, the asset/liabilities ratio l, and the population/liabilities ratio p. There is no known closed-form solution, so we solve the dynamic programming problem numerically through value function iteration. Details of the computational procedure are found in the Appendix.

4.1 Simulation Procedure

After we have found the solution to the dynamic programming problem, we use simulation to examine the bank's behavior over time. Starting at an initial state of $(m_0, l_0, p_0) = (1, 1, 1)$, we simulate 1000 independent runs with 1000 periods each, and observe the firm's state and choice variables. For time until bankruptcy, we record the average across runs. For m_t, l_t, and γ_t^L, we record the average across runs, of the time average in a run (before bankruptcy). We will perform the computational and simulation procedure for a range of environments in this paper; the parameter values we will use are not derived from empirical data, but chosen to reveal interesting behavior.

We present two examples to illustrate optimal bank behavior.

4.2 A simple example without lending

We use the following parameters:

- β (discount rate) = 0.9
- r^D (interest rate on deposits) = 0, r^L (interest rate on loans) = 0. Both interest rates are zero.
- α^L (fraction of loans repaid) = 0.4, implying an expected duration of 2.5.
- $\boldsymbol{\alpha_t^D}$ (fraction of deposits withdrawn) = 0.8
- γ^D (new deposits as a fraction of current population) = 0.8

The important parameters are r^D, r^L, and $\boldsymbol{\alpha_t^D}$, which in this case is a constant. There is no interest rate spread and loans do not offer a positive return. Intuitively, since the bank owners are impatient ($\beta < 1$) then it is optimal to not lend anything, since that would simply delay payouts into the future without a compensating return. The bank will pay out as much cash as possible while keeping the firm alive (up to a point). The computed optimal policy agrees with this intuition; the optimal choice of new loans is zero for all points in the grid space. This is analogous to a bank operating in runoff mode (i.e. not making new loans, but being repaid by old ones as they mature). The value of the firm comes from the current cash reserve, plus the stream of loan repayments which falls geometrically over time. Running the simulation procedure gives the following simulated observations (means across runs are reported, with standard deviations in parentheses):

- average m_t: 0.023 (0.000)
- average l_t: 0.028 (0.000)
- average γ_t^L: 0.000 (0.000)
- average lifetime: 87.000 (0.000)
- v: 1.796407

The bank, on average and conditional on survival, will hold as little cash as possible while still avoiding bankruptcy, which in this case turns out to be the smallest numeric value in our grid. Starting from period 1, the bank keeps the minimum level of cash to survive; there are no shocks to buffer against (and hence all the standard deviations are zero). It collects the repayment stream from the initial l_t until the benefit of looting the firm exceeds the benefit of continuing. Figure 2 shows the evolution of m_t and l_t over time; cash and asset levels drop monotonically as cash is extracted and the stock of past loans is drawn down without replenishment.

4.3 An example with lending

We use the following parameters:

- β (discount rate) = 0.9
- r^D (interest rate on deposits) = 0.05, r^L (interest rate on loans) = 0.13
- α^L (fraction of loans repaid) = 0.4, implying an expected duration of 2.5
- $\boldsymbol{\alpha_t^D}$ (fraction of deposits withdrawn) $\in \{0.7, 0.9\}$ with equal probability
- γ^D (new deposits as a fraction of current deposits) = 0.8

Now the interest rate spread is positive and there is uncertainty in withdrawals. The simulation results show:

- average m_t: 0.438 (0.002)
- average l_t: 4.300 (0.017)
- average γ_t^L: 0.287 (0.000)
- average lifetime: 999.000000 (0.000)
- v: 1.787359

On average, we observe that this bank holds about 43% of liabilities as cash, and keeps an asset/liability ratio of about 4.3. Average lending is about 29% of initial loans. The average lifetime is equal to the maximum number of simulated periods; every simulated bank survived until the end of the run. This implies that a bank following the optimal policy in this environment will never go bankrupt. Figure 3 plots one sample path of the bank's state variables for 100 periods. After the initial period, l eventually settles down to a stable range at around 4.3. We would not take the exact sample path too literally, since this may be dependent on the limits of our simulation; but we can be sure that this firm does not go bankrupt.

5 Comparative Statics

In this section we examine the effect of changes in the environment on optimal behavior through simulation. We systematically vary a subset of the parameters, solve the model, and simulate bank behavior over time. We are interested in parameters that affect bank viability, value, lending, and cash reserve.

5.1 Interest Rates

As a consistency check, we examine the effect of varying the deposit and loan interest rates. Obviously, higher loan rates and lower deposit rates should be better for the bank, all else equal. We use the following parameters:

- β (discount rate) = 0.9
- α^L (fraction of loans repaid) = 0.4, implying an expected duration of 2.5
- $\boldsymbol{\alpha_t^D}$ (fraction of deposits withdrawn) $\in \{0.7, 0.9\}$ with equal probability.
- γ^D (new deposits as a fraction of current deposits) = 0.8

For r^D, we use values from the set $\{0.03, 0.05, 0.1, 0.12\}$. For r^L, we use a range from 0.1 to 0.16.

Figures 4 and 5 plot mean lifetime and γ_t^L (new lending as a fraction of initial lending) over our range of parameters. For a given r^D, there is a threshold level of r^L below which lifetime is just a few periods. Above the threshold, lifetime is equal to the maximum number of simulated periods, indicating that the firm never goes bankrupt. In environments where the firm dies out quickly, lending is zero, as indicated by the choice variable γ_t^L. We interpret this as saying that for those environments, any amount of lending is unprofitable in expectation; therefore, optimal behavior is to not lend at all, as in our first numerical example. In these cases the bank is operating in "runoff mode" and surviving on the stock of past loans. This is dependent on the specific form of the probability distribution of withdrawals; we will later see a case where lending takes place, but firms can still go bankrupt.

We call environments where lending is unprofitable "nonviable". For our specification, parameter space is divided into a nonviable and viable region; as expected, a higher (lower) interest rate on loans (deposits) moves from the nonviable to the viable region. Figures 6, 7, and 8 show cash/liabilities ratio, assets/liabilities ratio and firm value. We would not take the values for viable environments literally; as the interest rate on loans increases, we run up against the limits of our simulation procedure, since the maximum value for l_t is bounded by our grid size. Nevertheless, we believe it is useful to show the general direction of change as the environment changes.

5.2 Uncertainty of Withdrawals

Here, we examine the effect of uncertainty in withdrawals (and hence, uncertainty in deposit duration) on cash reserves, investment, firm value, and bankruptcy risk. We model uncertainty in a simple way. α_t^D can take on two values, $\overline{\alpha^D} - \delta$ and $\overline{\alpha^D} + \delta$. We use the following parameters:

- β (discount rate) = 0.9
- r^L (interest rate on loans) = 0.13
- α^L (fraction of loans repaid) = 0.4, implying an expected duration of 2.5
- $\boldsymbol{\alpha_t^D}$ (fraction of deposits withdrawn) $\in \left\{\overline{\alpha^D} - \delta, \overline{\alpha^D} + \delta\right\}$ with equal probability.
- $\overline{\alpha^D} = 0.8$
- γ^D (new deposits as a fraction of current deposits) = 0.8

r^D takes on the values $\{0.02, 0.04, 0.06, 0.0\}$. For δ, we use a range from 0 to 0.17.

Figures 9 and 10 show mean lifetime and γ_t^L (new lending as a fraction of initial lending) over our range of parameters. As before, the parameter space is divided into a region with nonviable firms that go bankrupt in a few periods and do not lend, and viable firms that survive until the end of the simulation. As uncertainty in withdrawals increases, we cross from the viable region to the nonviable region, which makes intuitive sense; all else equal, increased uncertainty in withdrawals should result in a less profitable environment. For viable firms, the amount of lending is unaffected by uncertainty or r^D.

Figures 11 and 12 show the cash/liabilities and asset/liabilities ratio. For viable firms, both of these ratios decline as uncertainty increases, which at first glance seems counterintuitive. We cannot offer an explanation at this point; one possible reason is that the way we model uncertainty is not neutral with respect to the growth of deposits.

5.3 Uncertainty of Loan Repayments

So far, we have only modeled uncertainty in withdrawals. We have seen that parameter space, at least for the specifications used so far, seems to be divided into two regions; one with firms that make no attempt at survival (i.e. do not lend), and firms that never go bankrupt. We interpret this as saying that if withdrawals are the only source of uncertainty, then the level of uncertainty determines whether a particular environment is profitable for banking or not; but if it is profitable, then an optimally run bank (that starts with sufficient capital) will not go bankrupt, given our specification. We introduce a second source of uncertainty in loan repayments. As with deposits, we assume $\boldsymbol{\alpha_t^L}$ can take on two values, $\overline{\alpha^L} - \delta^L$ and $\overline{\alpha^L} + \delta^L$, with equal probability. We assume $\boldsymbol{\alpha_t^D}$ and $\boldsymbol{\alpha_t^L}$ are independent. This does not capture default risk, but rather prepayment risk, since the stock of loans is diminished by exactly the amount of repayment. We use the following parameters:

- β (discount rate) = 0.9
- r^L (interest rate on loans) = 0.13, r^D (interest rate on deposits) = 0
- α^L (fraction of loans repaid) = 0.4, implying an expected duration of 2.5
- $\boldsymbol{\alpha_t^D}$ (fraction of deposits withdrawn) $\in \left\{\overline{\alpha^D} - \delta, \overline{\alpha^D} + \delta\right\}$ with equal probability.
- $\overline{\alpha^D} = 0.8$
- $\boldsymbol{\alpha_t^L}$ (fraction of loans repaid) $\in \left\{\overline{\alpha^L} - \delta, \overline{\alpha^L} + \delta\right\}$ with equal probability.
- $\overline{\alpha^D} = 0.4$
- γ^D (new deposits as a fraction of current deposits) = 0.8

Both δ^L and δ^D take on a range of values from 0 to 0.2.

Figures 14 and 15 show mean lifetimes and lending across our range of parameters. In contrast to the previous two cases, mean lifetimes take on intermediate values between 0 and 999; mean lending also shows a range of values between 0 and the maximum level. This indicates that for some parameter values, a bank following the optimal policy has a nonzero probability of bankrupt in a finite number of periods; this is similar to the result proved in the "optimal dividends" problem that the firm goes bankrupt in finite time, almost surely. Increasing uncertainty, either in withdrawals or repayments, decreases firm value (Figure 18), mean lifetime, and lending.

Figures 19 and 20 show mean m and l values in parameter space. The points on the left represent firms that do not go bankrupt; points on the right represent firms that have finite lives. As before, firms that do not go bankrupt have decreasing cash/liabilities and asset/liabilities ratios as uncertainty in either withdrawals or deposits increases. However, for firms that have finite lives, the mean cash/liability ratio (observed before bankruptcy) now increases with uncertainty, as intuition would predict.

6 Discussion

The parameter values and probability distributions used in this paper were selected for simplicity and interesting behavior, not realism. The next step would be to use more realistic modeling assumptions, as well as comparing the model's predictions against empirical data.

In this paper the supply of deposits, and the demand for loans, is taken as exogenous. A natural extension is to endogenize supply and demand by modeling the decision problem of the bank's customers. We can imagine the bank offering an interest rate to borrowers and lenders, who choose to borrow or lend based on standard models of savings and portfolio choice. In a companion paper we attempt to do this for a simpler model of an inventory management firm, a Ponzi scheme (Carpio 2011).

7 Conclusion

In this paper, we introduce a dynamic model of a banking firm based on inventory management. The bank takes deposits and makes loans, which may have different maturities; it maximizes its value based on dividends and the amount of lending. The model offers a parsimonious explanation of dividend payouts, maturity mismatch, quantities of credit, and bankruptcy; mismatch arises non-strategically, from profit maximization. Our numerical experiments show that maturity mismatch can be optimal if the level of uncertainty in withdrawals is not too high. By itself, uncertainty in withdrawals does not necessarily cause an optimally behaving bank to fail, but it can when paired with uncertainty in repayments. This model should be viewed as a first step towards understanding how financial intermediaries work. Possible extensions include endogenizing the supply and demand of credit, more realistic parameter values, and applications to other types of financial intermediaries.

8 Appendix: Computational Procedure

We find the solution numerically through value function iteration. There are three state and two control variables. The state space for m, l, and p are discretized into a regularly spaced grid of 50, 40, and 30 points, respectively. The control space for x and γ_t^L are discretized into a grid of 40 points. The boundaries of the grids are: $m \in [0,4], l \in [0,5], p \in [0,2], x \in [0, m_t]$. The value function is represented as a function on the three-dimensional set of grid points, and trilinear interpolation is used to approximate function values between grid points. Discretization (rather than, say, polynomial approximation of the value function) is appropriate in this problem, since the value function can be non-differentiable and non-continuous. The computation-intensive code was implemented in C++ for performance, and takes advantage of multi-core processors, using Intel's Thread Building Blocks (TBB) library. Support code is implemented in Python for ease of development and debugging. To speed up convergence, we implemented multi-grids and when iterating through a sequence of parameter values, we save the current optimal value function and use it as the initial guess for the next iteration. The initial version of the code was heavily influenced by Stachurski (2009).

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Figure 1: Timeline of model

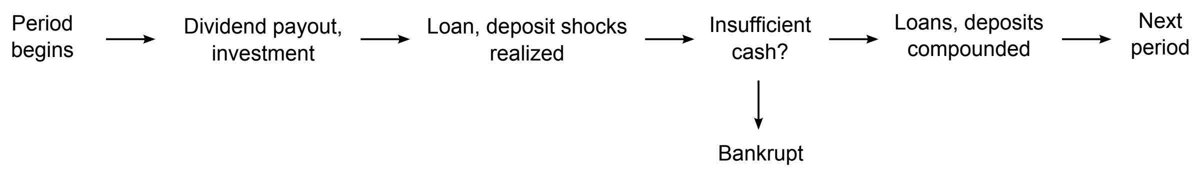

Figure 2: m_t, l_t over time (no-lending case)

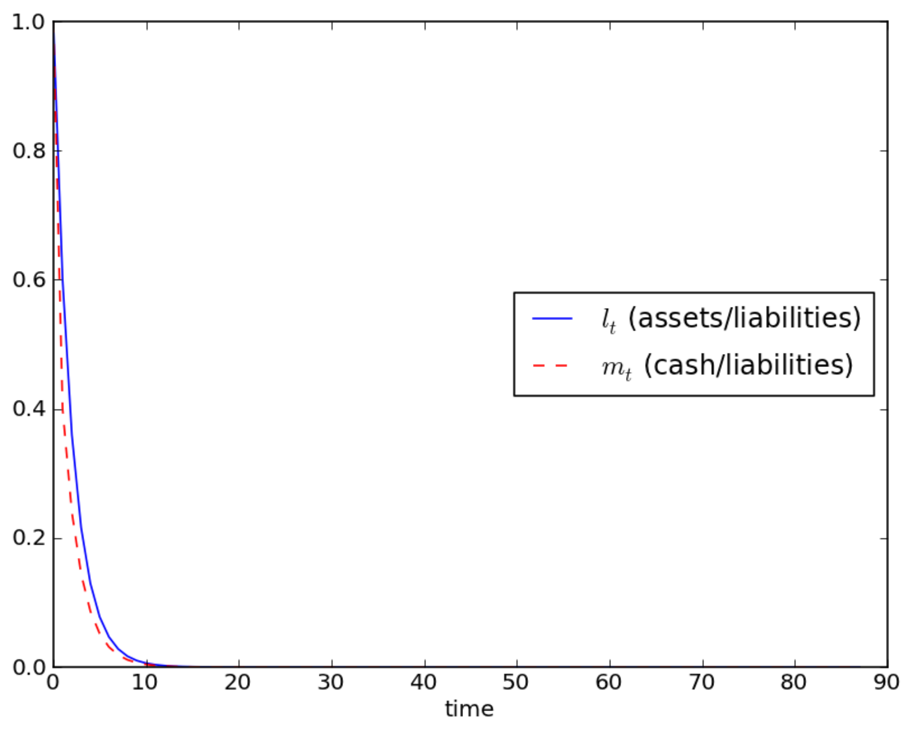

Figure 3: Sample path of m_t, l_t, γ_t^L over time (positive lending case)

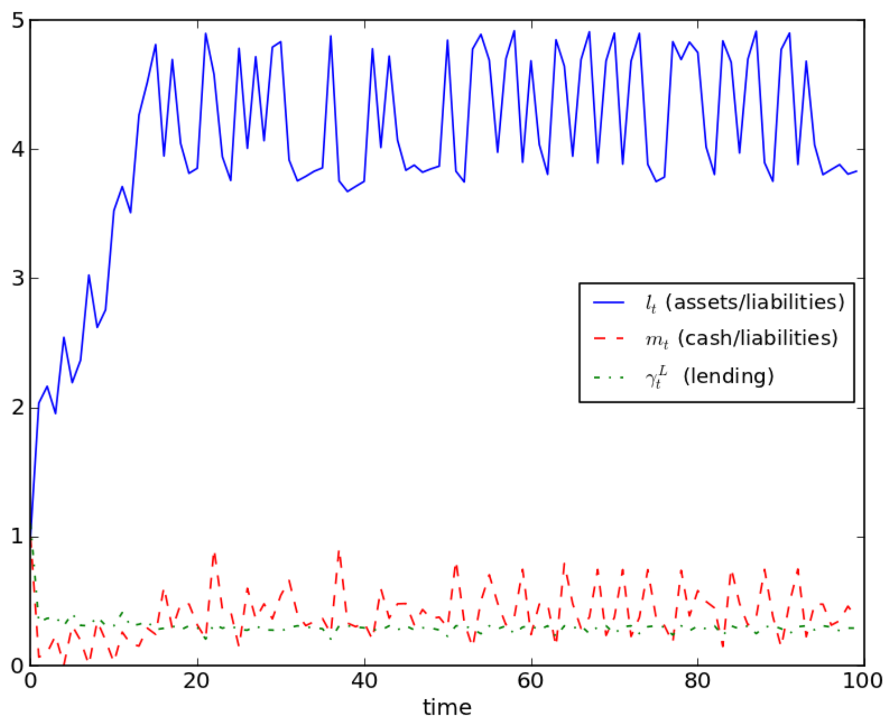

Figure 4: Mean bank lifetime vs. interest rate on loans, deposits

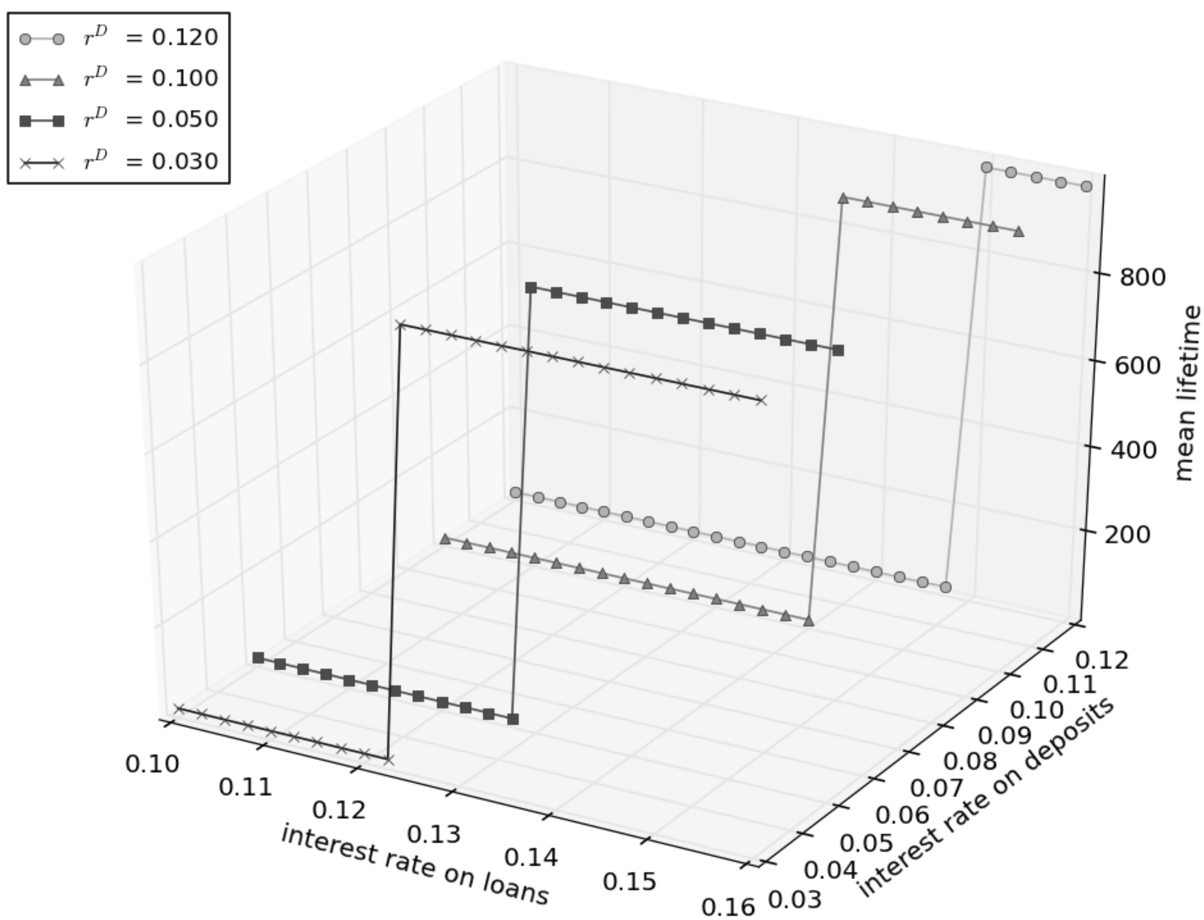

Figure 5: New lending as fraction of l_t vs. interest rate on loans, deposits

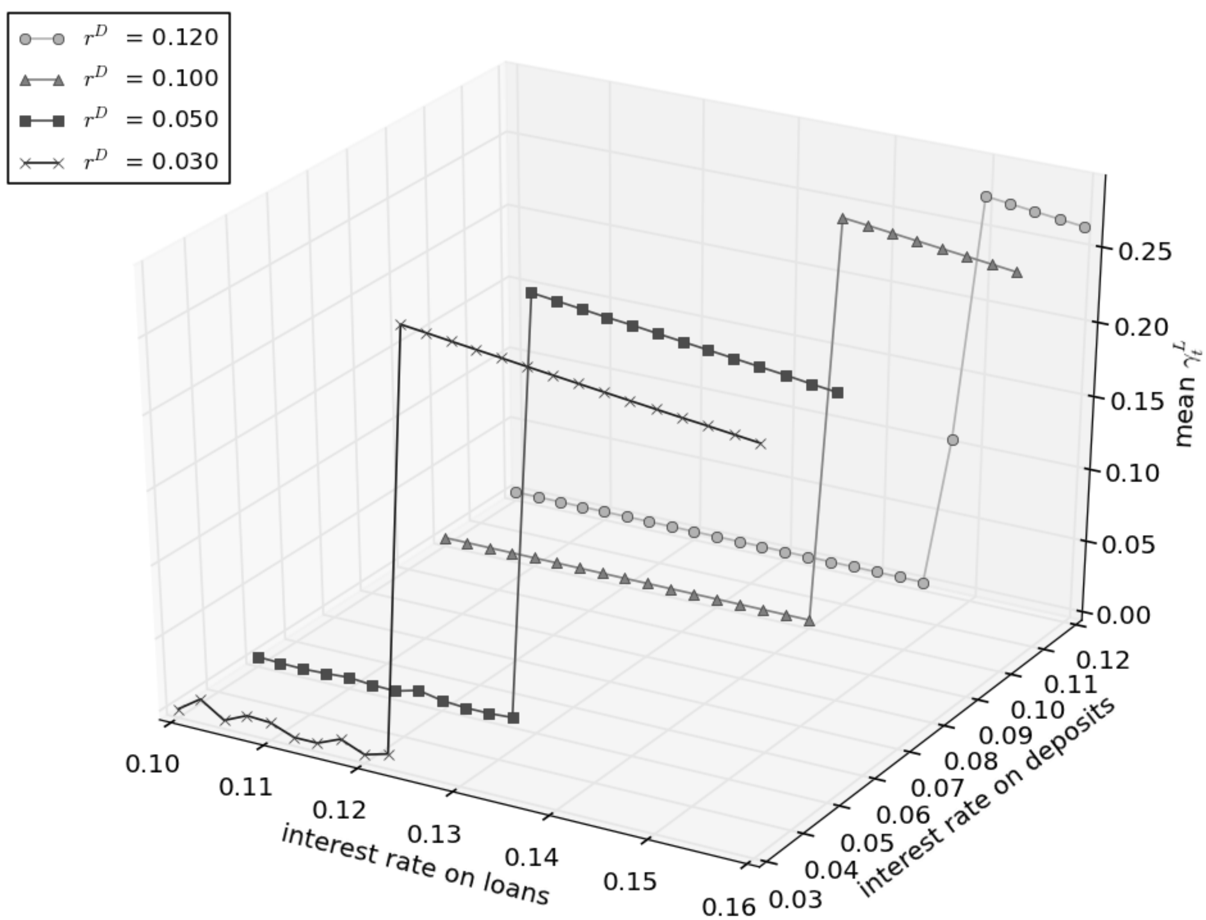

Figure 6: Mean cash/liabilities ratio vs. interest rate on loans

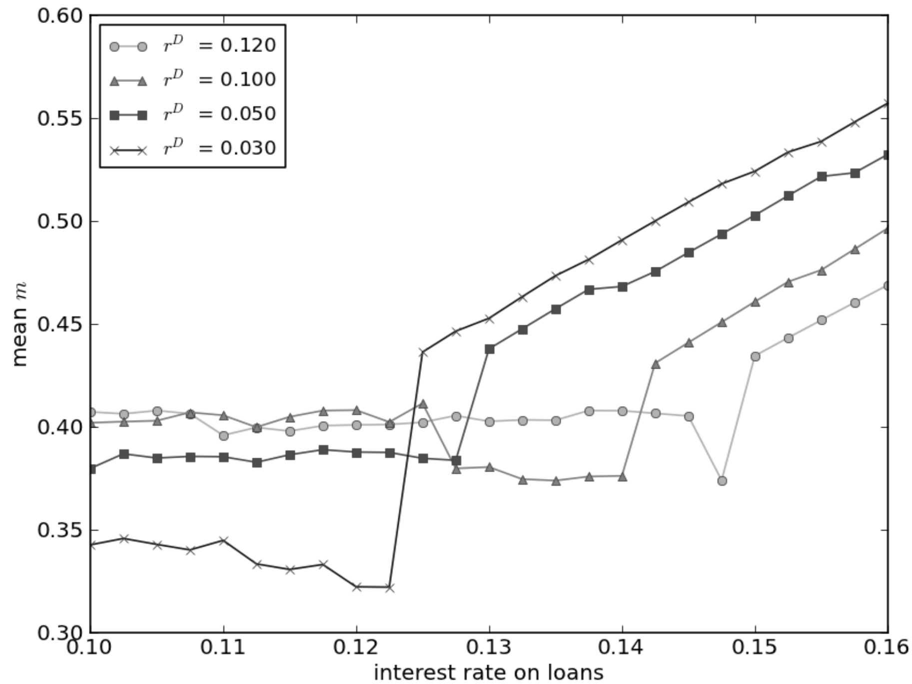

Figure 7: Mean assets/liabilities ratio vs. interest rate on loans

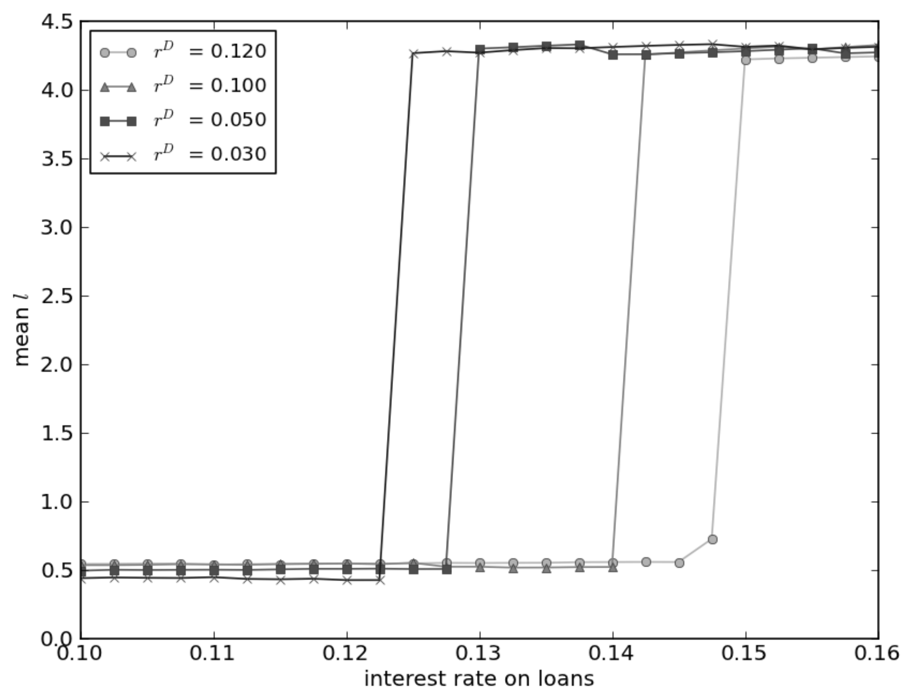

Figure 8: Firm value vs. interest rate on loans

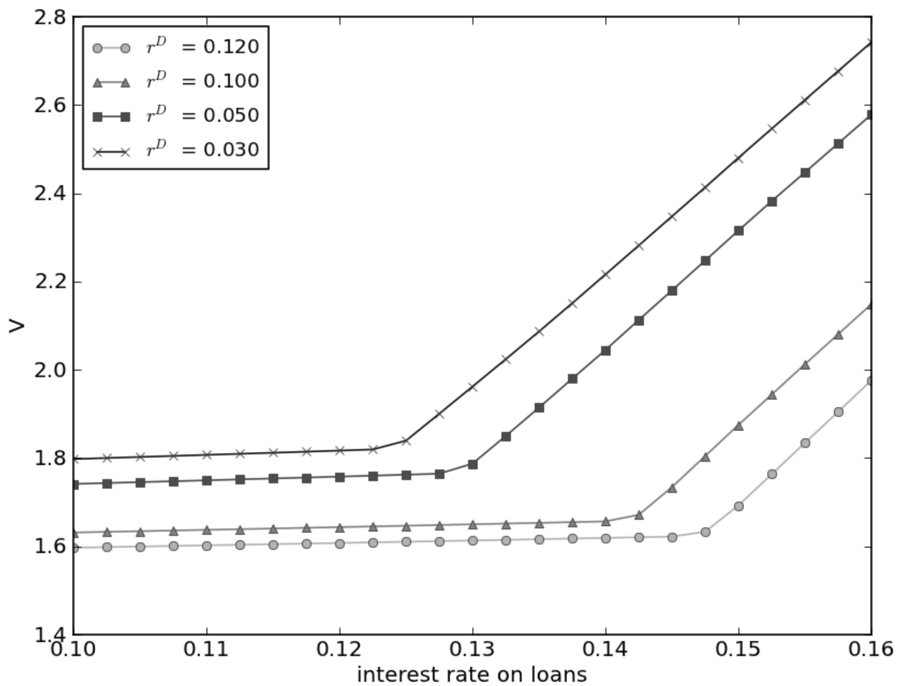

Figure 9: Mean bank lifetime vs. uncertainty in withdrawals, interest rate on deposits

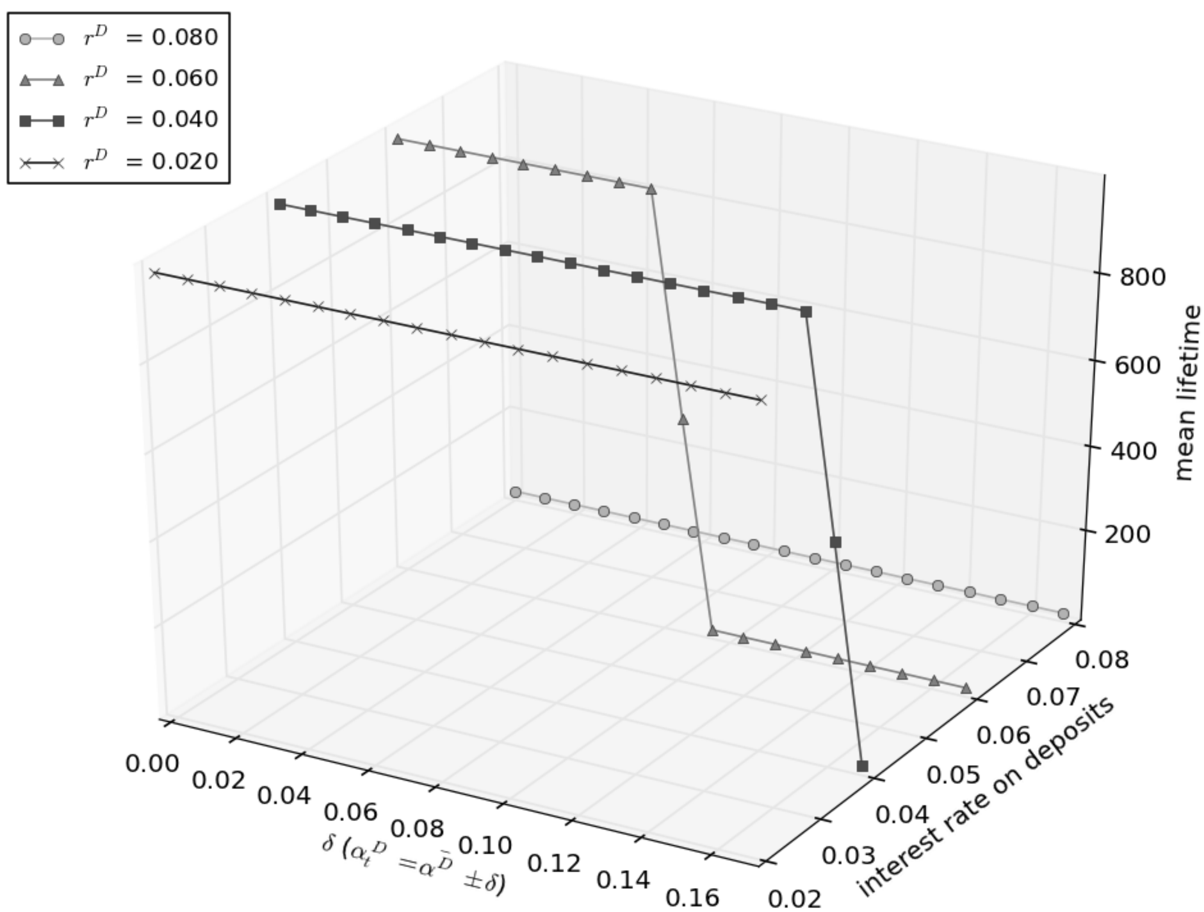

Figure 10: New lending as fraction of l_t vs. uncertainty in withdrawals, interest rate on deposits

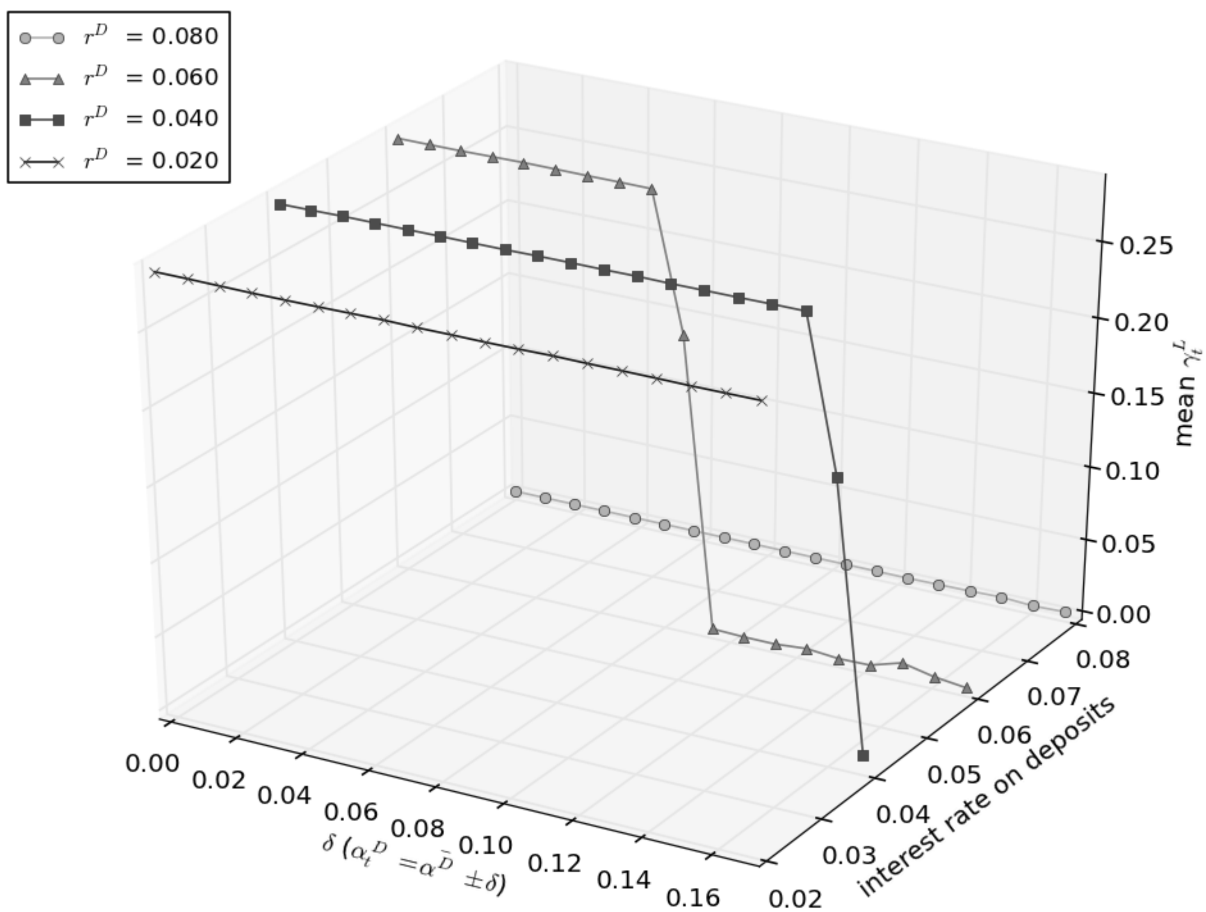

Figure 11: Mean cash/liabilities ratio vs. uncertainty in withdrawals

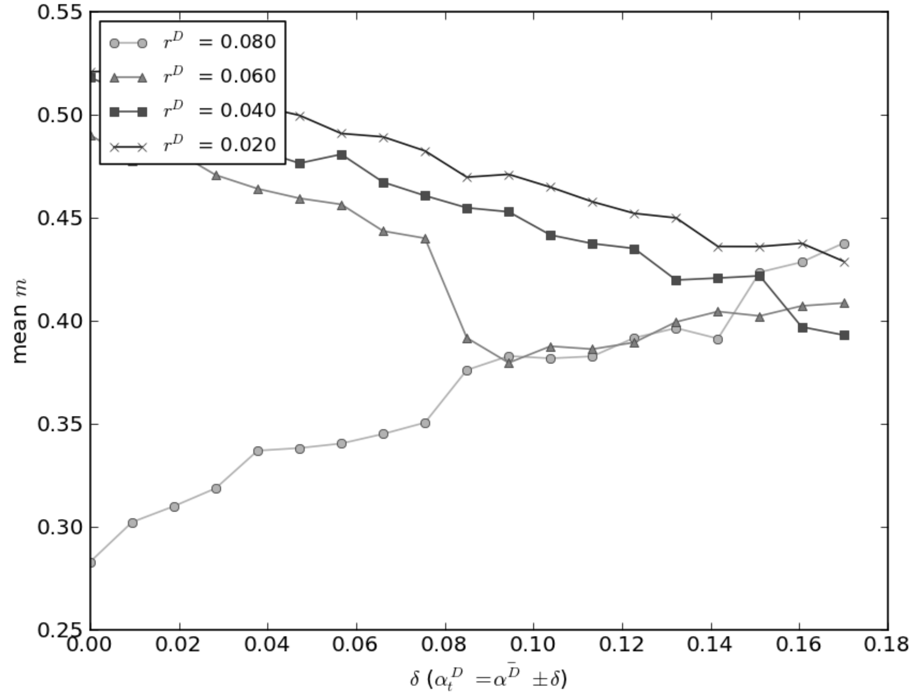

Figure 12: Mean assets/liabilities ratio vs. uncertainty in withdrawals

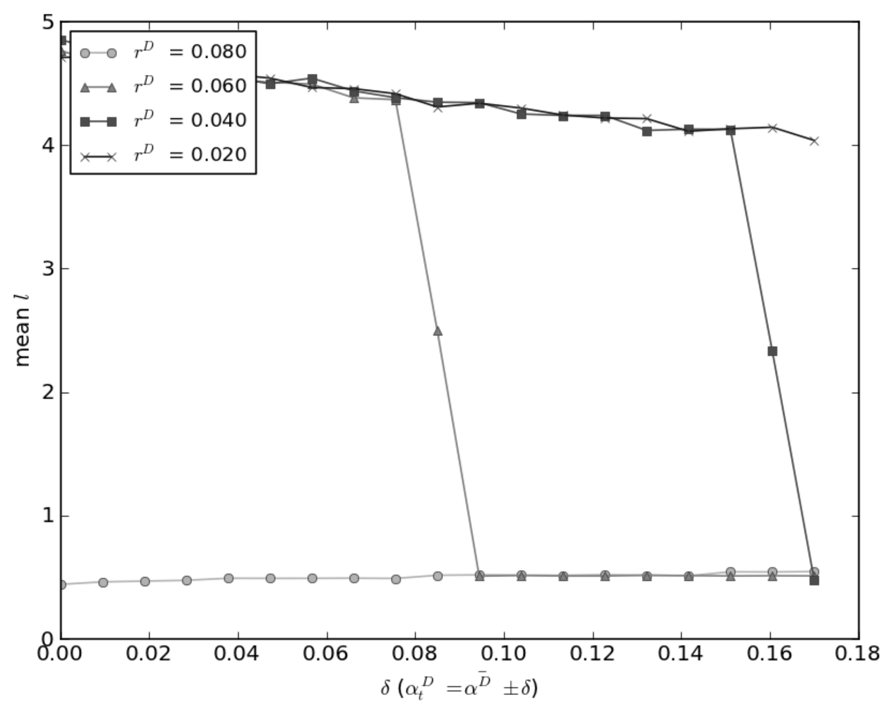

Figure 13: Firm value vs. uncertainty in withdrawals

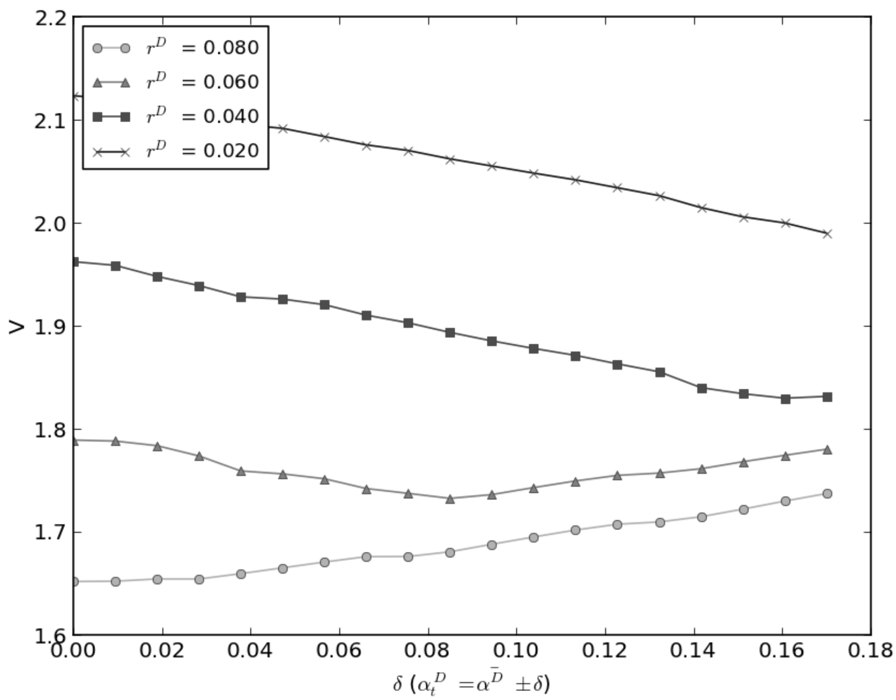

Figure 14: Mean bank lifetime vs. uncertainty in repayments, withdrawals

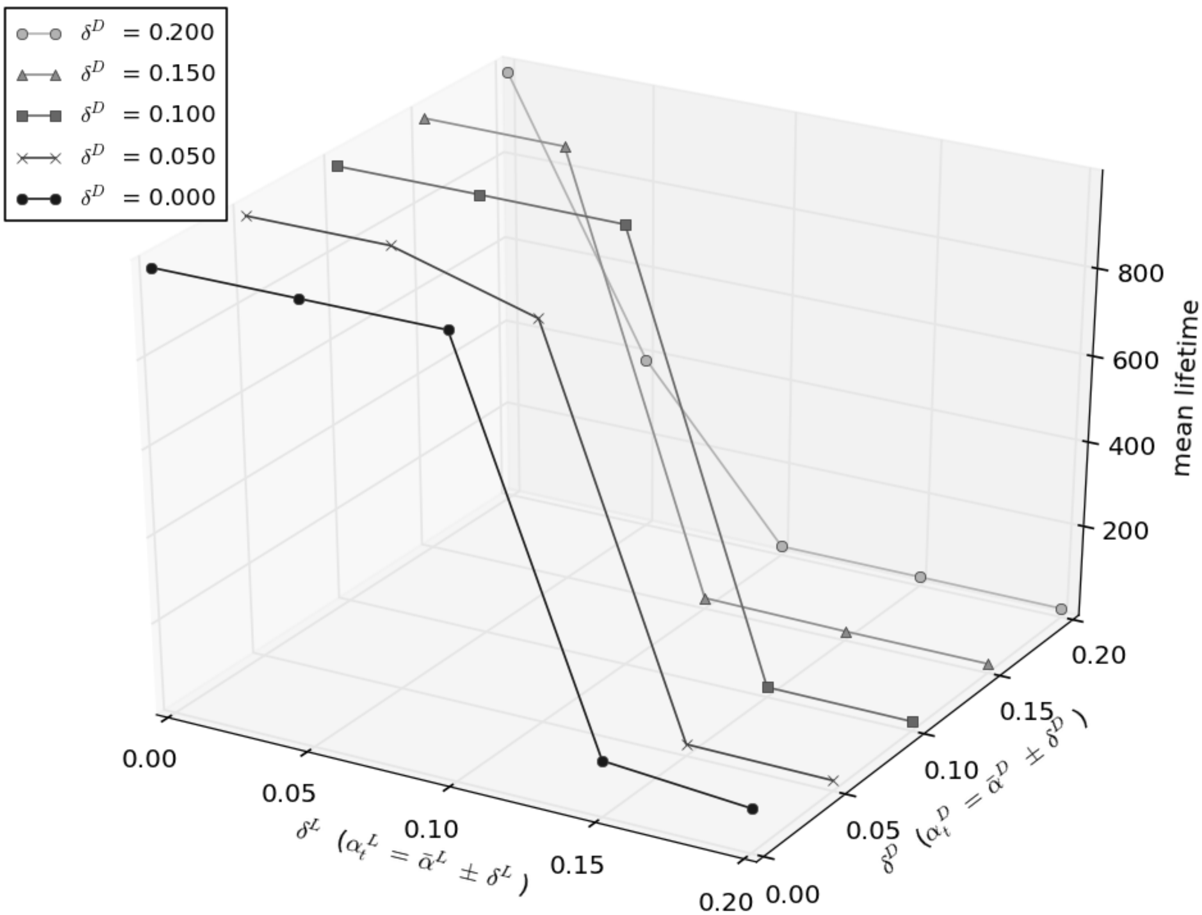

Figure 15: New lending as fraction of l_t vs. uncertainty in repayments, withdrawals

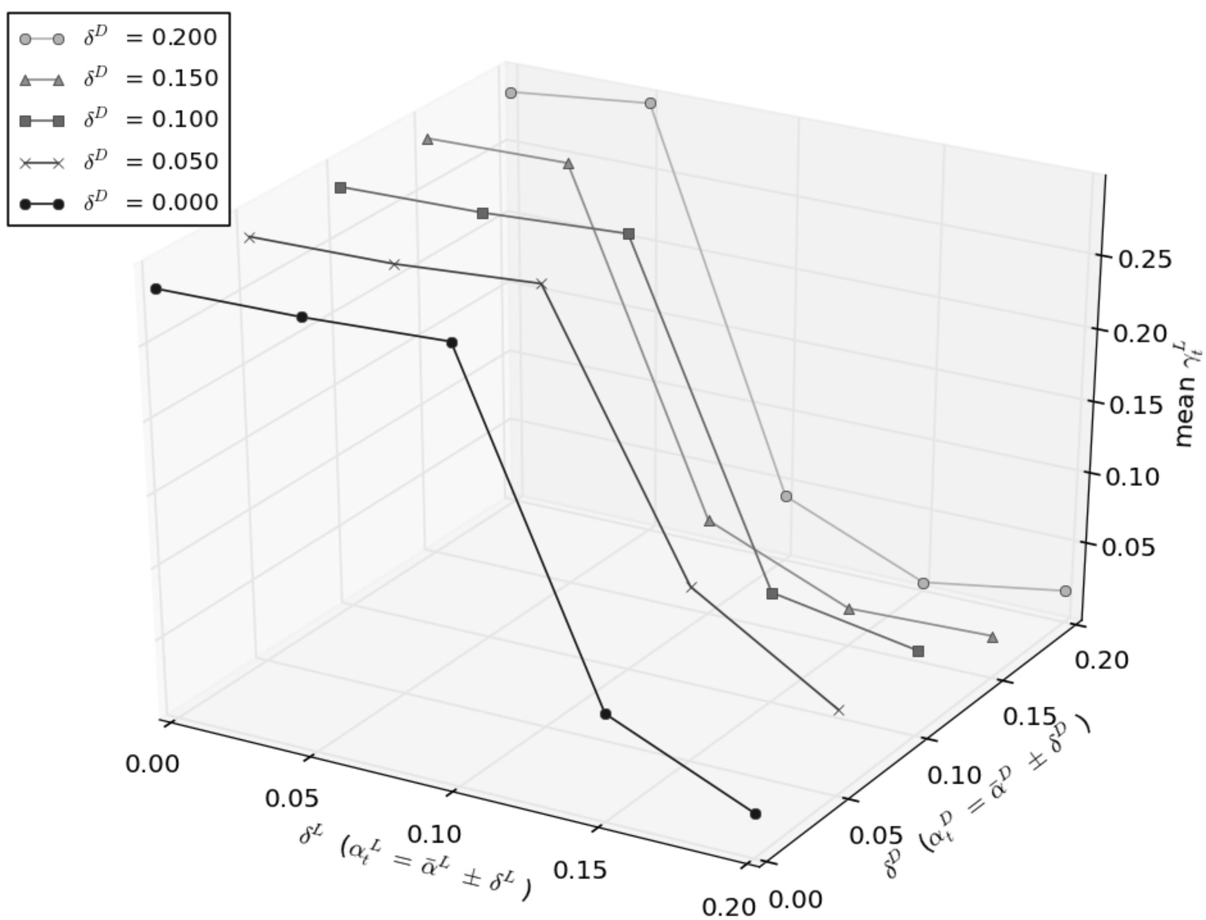

Figure 16: Mean cash/liabilities ratio vs. uncertainty in repayments

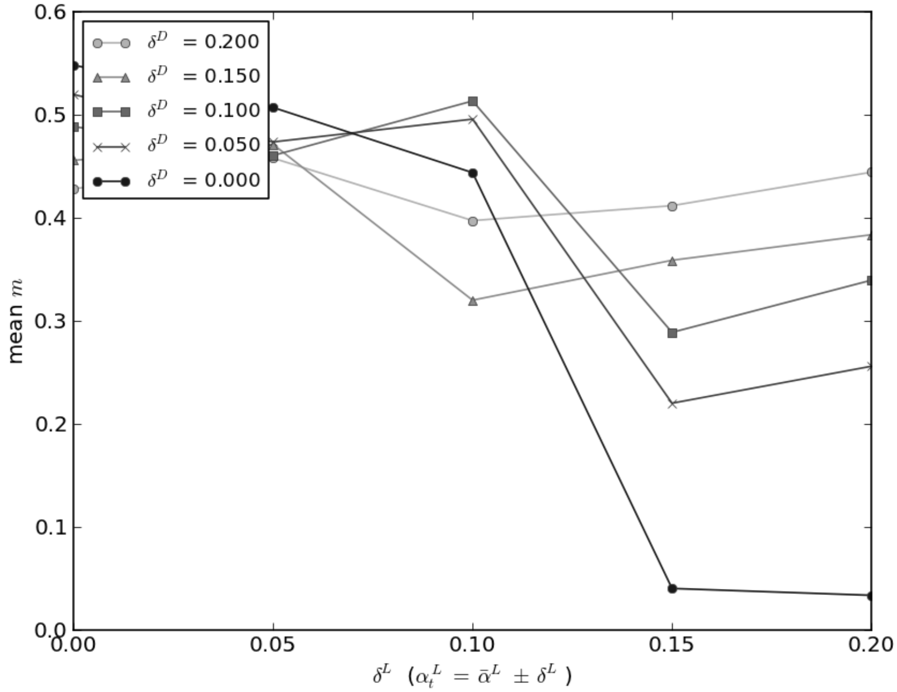

Figure 17: Mean assets/liabilities ratio vs. uncertainty in repayments

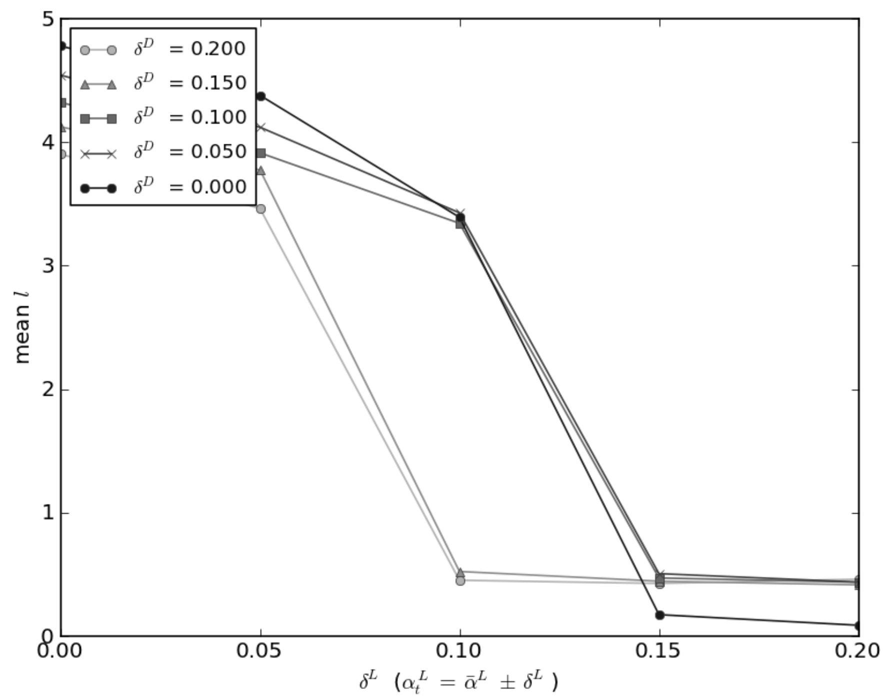

Figure 18: Firm value vs. uncertainty in repayments

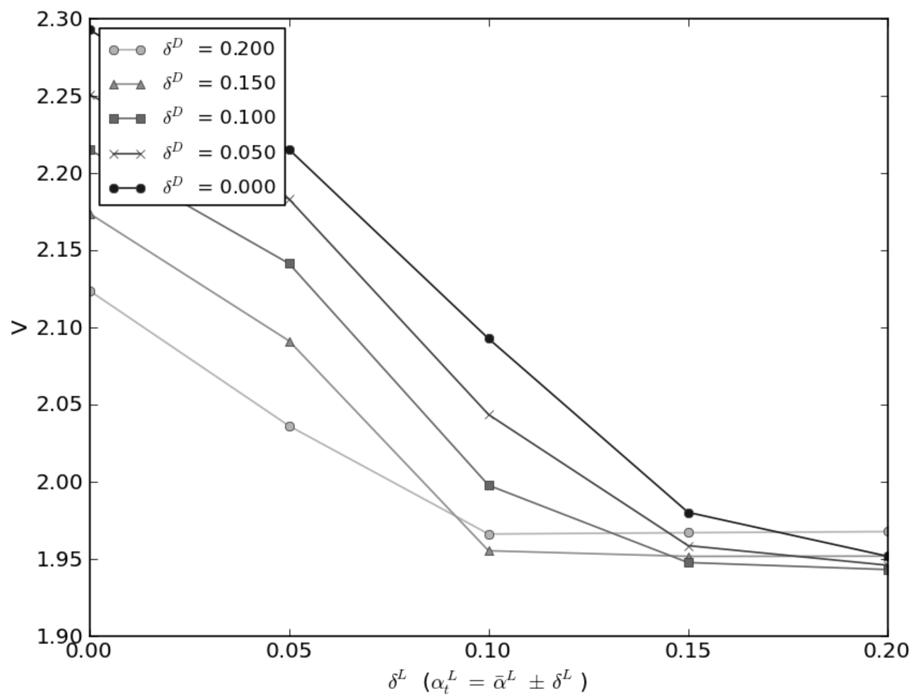

Figure 19: Mean cash/liabilities ratio vs. uncertainty in repayments, withdrawals

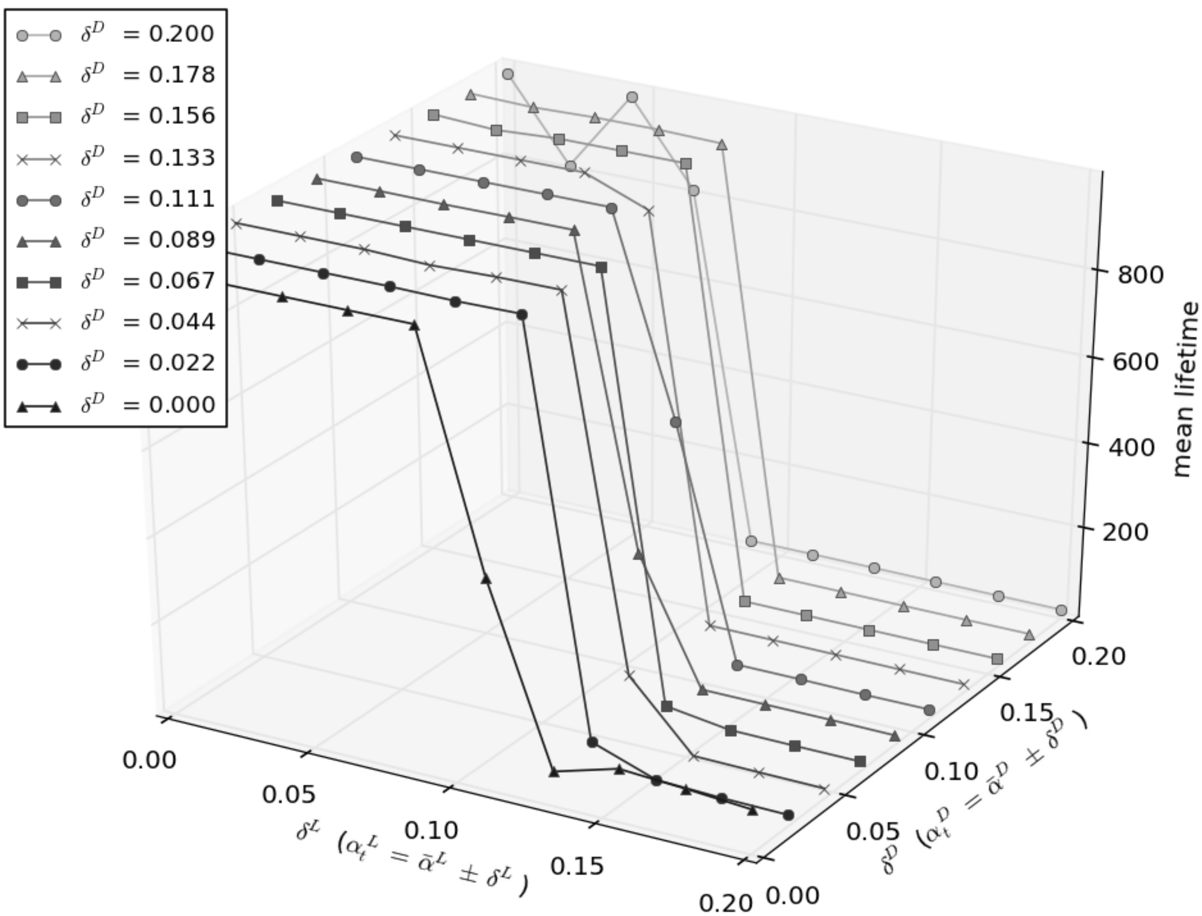

Figure 20: Mean assets/liabilities ratio vs. uncertainty in repayments, withdrawals