Advanced Microeconomic Analysis, Lecture 11

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Administrative Stuff

- The final exam will be on June 23 from 13:30-15:30 in Boxue 504. It will cover the material after the midterm exam.
- HW #4 is due today at the end of class. I will post the solutions and the last homework on the class web site.
- There will be no class next Monday, June 5.
- The last class will be June 12.

Review of Last Week

- A *mixed strategy* is a probability distribution over a player's strategy set.
- We can think of it as the player randomizing his choice according to a chosen probability distribution.
- A *pure strategy* is a mixed strategy with 100% probability on a single strategy.
- The payoff to the player is now the expected value of the payoff to pure strategies.
- A *mixed strategy Nash equilibrium* is a mixed strategy profile, such that no player has an incentive to change his mixed strategy.

Review of Last Week

- Any finite game is guaranteed to have at least one mixed strategy NE.
- Games that have no pure strategy NE (such as Football Penalty Kick) will have a MSNE.
- A mixed strategy profile $(M_1, ..., M_n)$ is a mixed strategy NE if and only if:
  1. For each player i, the expected payoff to every pure strategy with positive probability in M_i is equal; and
  2. the expected payoff to every pure strategy with zero probability in M_i is less than the value in the previous case.

Review of Last Week

- Review of probability: suppose we have a random experiment with a finite number of outcomes, or states.
- Let U denote the *universe*, the set of all possible outcomes.
- Each outcome has a given probability of occurring, and the sum of these probabilities must add up to 1.
- An *event*, A is a set of outcomes.
- If the outcome of the random experiment happens to be a member of the set A, we say "event A has occurred".

Review of Last Week

- The *conditional probability* of event A given that event B has occurred, is denoted $P(A|B)$.
- It is the probability that the outcome is in set A, given that we know it is in set B.
- Bayes' Theorem: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- We can rearrange it to get $P(B)P(A|B) = P(A \cap B) = P(A)P(B|A)$.
- This lets us calculate the conditional probability of any event, given any other event.

Review of Last Week

- For example: suppose we roll a fair, 4-sided die.
  - $U = \{1, 2, 3, 4\}$, with $P(1) = P(2) = P(3) = P(4) = 0.25$
  - Let $A = \{1, 2\}, B = \{2, 4\}$.
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(2)}{P(\{2,4\})} = \frac{0.25}{0.5} = \frac{1}{2}$
  - $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(2)}{P(\{1,2\})} = \frac{0.25}{0.5} = \frac{1}{2}$
- Suppose the die is not fair, and $P(1) = P(2) = P(3) = 0.2, P(4) = 0.4$.
  - $P(A|B) = \frac{P(2)}{P(\{2,4\})} = \frac{0.2}{0.6} = \frac{1}{3}$
  - $P(B|A) = \frac{P(2)}{P(\{1,2\})} = \frac{0.2}{0.4} = \frac{1}{2}$

Incomplete Information

- So far, we have assumed that players are perfectly informed about the payoffs of all other players.
- However, in many real-life situations, there is uncertainty about the opponents' payoffs. For example:
  - When you buy an item from a seller, the seller knows the item's quality, but you do not.
  - When two people get into a competition, each person knows his own strength, but not the other person's.
- We will show how to specify this situation as a strategic form game by adding two additional elements.

Player Types

- First, for each player i, we introduce a finite set of "types", T_i, that the player might be.
- For example:
  - A firm might have two types, "low production cost" and "high production cost".
  - A competitor might have two types, "strong" and "weak".
- A player's payoffs for a given joint pure strategy now also depend on his type.
- Let $T = \times_{i=1}^{N} T_i$, the set of joint types.
- Player i's payoff function u_i maps $S \times T$ to a real number.

Beliefs as Probability Distributions over Types

- Second, each player has *beliefs* about what all other players' type may be.
- A belief is a probability distribution over the set of possible types.
- For example, suppose there are 2 players, and Player 2 has two possible types: "weak" and "strong".
- Player 1 has a belief about Player 2's type, $(p, 1-p)$, where p is Player 1's *subjective probability* that Player 2 is "weak".
  - If $p = 0$, then Player 1 is certain that Player 2 is "strong".
  - If $p = 0.5$, then Player 1 thinks that it is equally likely that Player 2 is "weak" or "strong".
  - If $p = 1$, then Player 1 is certain that Player 2 is "weak".
- If Player 1 also has more than one type, we need to specify (possibly) different beliefs about Player 2, for each of Player 1's type.

Game of Incomplete Information (Bayesian Game)

- **Def 7.10**: A game of incomplete information (also called a Bayesian game) is a tuple $G = (p_i, T_i, S_i, u_i)_{i=1}^N$, where for each player i, the set of types T_i is finite, $u_i : S \times T \to \mathbb{R}$, and for each $t_i \in T_i, p_i(\cdot|t_i)$ is a probability distribution on T_{-i}.
- Here, $p_i(t_{-i}|t_i)$ is the probability that the players aside from i have joint type t_{-i}, conditional on Player i's own type being t_i.
- This allows for the possibility that Player i's own type is not independent of the other players' types: if Player i knows he is a specific type, this may give him more information on the distribution of other players' types.

The Associated Strategic Form Game

- Let $G = (p_i, T_i, S_i, u_i)_{i=1}^N$ be a game of incomplete information.
- We will construct a strategic form game G^* in which each player type in G is a separate player.
- For each player $i \in \{1, ..., N\}$ and each Player i-type $t_i \in T_i$, let t_i be a player in G^* whose finite set of pure strategies is S_i.
- Let $s_i(t_i) \in S_i$ denote the a pure strategy chosen by player $t_i \in T_i$.
- The payoff to player t_i from the joint pure strategy $s^* = (s_1(t_1), ..., s_N(t_N))$ is:

$$v_{t_i}(s^*) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(s_1(t_1), ..., s_N(t_N), t_1, ..., t_N)$$

The Associated Strategic Form Game

- In short: to construct the associated strategic form of a game of incomplete information:
  - Suppose Player i has multiple types. For each type j, create a new player, with the same set of pure strategies as Player i - Type j.
  - For each new player, their payoffs will be their *expected* payoffs using their beliefs over player types as the probability distribution.
- The number of players in this associated game will be the total number of types of all players in the original game of incomplete information.
- Then, a NE of the associated game is called a *Bayesian-Nash* equilibrium of the original game.

Bayesian-Nash Equilibrium

- A *Bayesian-Nash equilibrium* of a game of incomplete information is a Nash equilibrium of the associated strategic form game.
- By the existence of Nash equilibrium in finite strategic form games, every finite incomplete information game has at least one Bayesian-Nash equilibrium.

Example: A Coordination Game with Different Types

- Consider the coordination game we saw earlier, but now suppose that Player 2 can have two different types.
- Type-1 of Player 2 is the same as before, and prefers to choose the same strategy as Player 1.
- Type-2 of Player 2, on the other hand, prefers to choose a different strategy as Player 1.
- If Player 2 is Type-1, then the payoff matrix is:

	B	S
B	2,1	0,0
S	0,0	1,2

Example: A Coordination Game with Different Types

	B	S
B	2,0	0,2
S	0,1	1,0

- If Player 2 is Type-2, then the payoff matrix is as above.
- Let $p_1(t_1)$ denote Player 1's subjective probability that Player 2 is of Type-1. Then, $p_1(t_2) = 1 - p_1(t_1)$.
- We will assume that all players know their own types with certainty.
- Suppose $p_1(t_1) = p_1(t_2) = 0.5$.
- We will only look at pure strategies for what follows. However, Player 1 still uses expected payoffs, where the uncertainty now comes from the type of Player 2.

Type 1

	B	S
B	2,1	0,0
S	0,0	1,2

Type 2

	B	S
B	2,0	0,2
S	0,1	1,0

- We will create the associated strategic game, which has three players: Player 1 of the original game, Player 2 - Type 1, and Player 2 - Type 2.
- We will denote a joint strategy as e.g. $(B,(B,S))$, where the first element is Player 1's strategy, and the second element is (Type 1's strategy, Type 2's strategy).
- Player 1 has beliefs over the type of Player 2, so we need to calculate his expected payoffs.
- Player 2-Type 1 and Player 2-Type 2 do not need beliefs, since Player 1 has only one type, so we don't need to calculate their expected payoffs.

Type 1

	B	S
B	2,1	0,0
S	0,0	1,2

Type 2

	B	S
B	2,0	0,2
S	0,1	1,0

- Player 1's expected payoff to each joint strategy is:
  - $(B,(B,B))$: 0.5*2 + 0.5*2 = 2
  - $(B,(B,S))$: 0.5*2 + 0.5*0 = 1
  - $(B,(S,B))$: 0.5*0 + 0.5*2 = 1
  - $(B,(S,S))$: 0.5*0 + 0.5*0 = 0
  - $(S,(B,B))$: 0.5*0 + 0.5*0 = 0
  - $(S,(B,S))$: 0.5*2 + 0.5*1 = 0.5
  - $(S,(S,B))$: 0.5*1 + 0.5*0 = 0.5
  - $(S,(S,S))$: 0.5*1 + 0.5*1 = 1
- Let's convert this to a 3-player game.

Type 2 chooses B

	B	S
B	2,_,_	1,_,_
S	0,_,_	0.5,_,_

Type 2 chooses S

	B	S
B	1,_,_	0,_,_
S	0.5,_,_	1,_,_

- In each matrix, the row player is Player 1, and the column player is Player 2 - Type 1.
- The left matrix is when Type 2 chooses B; the right matrix is when Type 2 chooses S.
- First, let's fill in Player 1's payoffs from the expected payoffs below.
  - $(B,(B,B))$: 0.5*2 + 0.5*2 = 2
  - $(B,(B,S))$: 0.5*2 + 0.5*0 = 1
  - $(B,(S,B))$: 0.5*0 + 0.5*2 = 1
  - $(B,(S,S))$: 0.5*0 + 0.5*0 = 0
  - $(S,(B,B))$: 0.5*0 + 0.5*0 = 0
  - $(S,(B,S))$: 0.5*2 + 0.5*1 = 0.5
  - $(S,(S,B))$: 0.5*1 + 0.5*0 = 0.5
  - $(S,(S,S))$: 0.5*1 + 0.5*1 = 1

Type 2 chooses B

	B	S
B	2,1,_	1,0,_
S	0,0,_	0.5,2,_

Type 2 chooses S

	B	S
B	1,1,_	0,0,_
S	0.5,0,_	1,2,_

- Then, let's fill in Player 2 (Type 1)'s payoffs from the matrix below.

Type 1

	B	S
B	2,1	0,0
S	0,0	1,2

Type 2 chooses B

	B	S
B	2,1,0	1,0,0
S	0,0,1	0.5,2,1

Type 2 chooses S

	B	S
B	1,1,2	0,0,2
S	0.5,0,0	1,2,0

- Finally, let's fill in Player 2 (Type 2)'s payoffs from the matrix below.

Type 2

	B	S
B	2,0	0,2
S	0,1	1,0

Example: A Coordination Game with Different Types

- We will solve this as a three-player game. We can compute the best response functions and see if there is an intersection.
- The best responses of Player 1 are:
$B_1(B,B) = B, B_1(B,S) = B, B_1(S,B) = B, B_1(S,S) = S$.
- The best responses of both types, only depend on Player 1's strategy.
  - Type 1: $B_{21}(B) = B, B_{21}(S) = S$
  - Type 2: $B_{22}(B) = S, B_{22}(S) = B$
- $(B,(B,S))$ is the unique Nash equilibrium, since all players are playing best responses to the other players.
- In this equilibrium, Player 1 plays B, Player 2-Type 1 plays B, and Player 2-Type 2 plays S.

Example: A Coordination Game with Different Types

- In this equilibrium, Player 1 plays B, Player 2-Type 1 plays B, and Player 2-Type 2 plays S.
- Player 1 does not know which type his opponent is, but has the following belief:
  - with probability 0.5, his opponent is Type 1
  - with probability 0.5, his opponent is Type 2
- Player 1 plays B, with an expected payoff of $0.5*2 + 0.5*0 = 1$.
- For Player 2, there is no uncertainty, since he knows his type.
- If Player 2 is a Type 1, he will play B and get a certain payoff of 1.
- If Player 2 is a Type 2, he will play S and get a certain payoff of 2.

Extensive Form Games (Chapter 7.3)

- So far, we've been using strategic form (or normal form) games. All players are assumed to move simultaneously.
- This cannot capture a sequential situation, where one player moves, then another...
- Or, if one player can get information on the moves of the other players, before making his own move.
- We will introduce a way of specifying a game that allows this.

Example: An Entry Game

- Suppose we have a situation where there is an *incumbent* and a *challenger*.
- For example, an industry might have an established dominant firm.
- A challenger firm is deciding whether it wants to enter this industry and compete with the incumbent.
- If the challenger enters, the incumbent chooses whether to engage in intense (and possibly costly) competition, or to accept the challenger's entry.

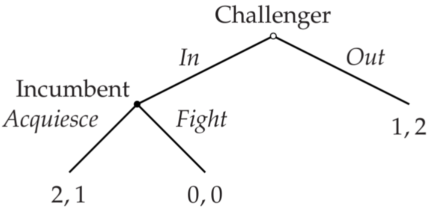
Entry Game

- There are two players: the incumbent and the challenger.
- We will refer to the set of possible choices at each node as a set of *actions*.
- A *strategy* for player i specifies a particular action to be played at each of player i's nodes.
- The challenger moves first, has two actions: *In* and *Out*.
- If the challenger chooses *In*, the incumbent chooses *Fight* or *Acquiesce*.
- Challenger's preference over outcomes: $(In, Acquiesce) > (Out) > (In, Fight)$
- Incumbent's preference over outcomes: $(Out) > (In, Acquiesce) > (In, Fight)$
- We can represent these preferences with the payoff functions (challenger is u_1):

$$u_1(In, Acquiesce) = 2, u_1(Out) = 1, u_1(In, Fight) = 0$$

$$u_2(Out) = 2, u_2(In, Acquiesce) = 1, u_2(In, Fight) = 0$$

Game Tree



- We can represent this game with a tree diagram.
- The root node of the tree is the first move in the game (here, by the challenger).
- Each action at a node corresponds to a branch in the tree.
- Outcomes are leaf nodes (i.e. there are no more branches).
- The first number at each outcome is the payoff to the first player (the challenger).

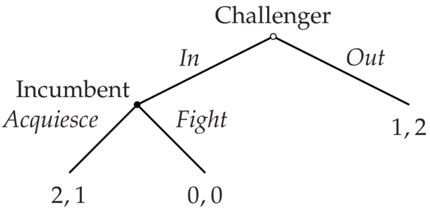
Formal Specification of an Extensive Game

- Formally, we need to specify all possible sequences of actions, and all possible outcomes.
- A *history* is the sequence of actions played from the beginning, up to some point in the game.
  - In the tree, a history is a path from the root to some node in the tree.
  - In the entry game, all possible histories are: $\varnothing$ (i.e. at the beginning, no actions played yet), $(In), (Out), (In, Acquiesce), (In, Fight)$.
- A *terminal history* is a sequence of actions that specifies an outcome, which is what players have preferences over.
  - In the tree, a terminal history is a path from the root to a leaf node (a node with no branches).
  - In the entry game, the terminal histories are: $(Out), (In, Acquiesce), (In, Fight)$.
- A *player function* specifies whose turn it is to move, at every non-terminal history (every non-leaf node in the tree).

Formal Specification of an Extensive Game

- An extensive game is specified by four components:
    - A set of **players**
    - A set of **terminal histories**, with the property that no terminal history can be a subsequence of some other terminal history
    - A **player function** that assigns a player to every non-terminal history
    - For each player, **preferences** over the set of terminal histories
- The sequence of moves and the set of actions at each node are implicitly determined by these components.
- In practice, we will use trees to specify extensive games.

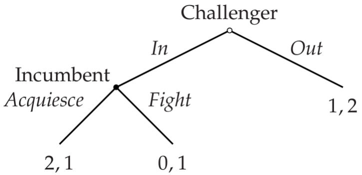
Solutions to Entry Game



- How can we find the solution to this game?
- First approach: Each player will imagine what will happen in future nodes, and use that to determine his choice in current nodes.
- Suppose we're at the node just after the challenger plays *In*.
- At this point, the payoff-maximizing choice for the incumbent is *Acquiesce*, which gives a payoff pair (2,1).
- So, at the beginning, the challenger might assume playing *In* gives a payoff pair of (2,1), which gives a higher payoff than *Out*.
- This approach is called *backwards induction*: imagining what will happen at the end, and using that to determine what to do in earlier situations.

Backwards Induction

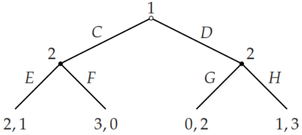
- At each move, for each action, a player deduces the actions that all players will rationally take in the future.
- This gives the outcome that will occur (assuming everyone behaves rationally), and therefore gives the payoff to each current action.
- However, in some cases, backwards induction doesn't give a clear prediction about what will happen.



- In this version of the Entry Game, both *Acquiesce*, *Fight* give the same payoff to the incumbent. Unclear what to believe at the beginning of the game.
- Also, games with infinitely long histories (e.g. an infinitely repeating game).

Strategies in Extensive Form Games

- Another approach is to formulate this as a strategic game, then use the Nash equilibrium solution concept.
- We need to expand the action sets of the players to take into account the different actions at each node.
- For each player i, we will specify the action chosen at all of i's nodes, i.e. every history after which it's i's turn to move
- **Definition**: A **strategy** of player i in an extensive game with perfect information is a function that assigns to each history h after which it is i's turn to move, an action in $A(h)$ (the actions available after h).



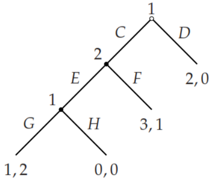
- In this game, Player 1 only moves at the start (i.e. after the empty history $\varnothing$). The actions available are C, D, so Player 1 has two strategies: $\varnothing \to C, \varnothing \to D$.
- Player 2 moves after the history C and also after D. After C, available actions are E, F. After D, available actions are G, H.
- Player 2 has four strategies:

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H

- In this case, it's simple enough to write them together. We can refer to these strategies as EG, EH, FG, FH. The first action corresponds to the first history C.

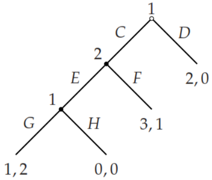
Strategies in Extensive Form Games

- We can think of a strategy as an action plan or contingency plan: If Player 1 chooses action X, do Y.
- However, a strategy must specify an action for *all* histories, even if they do not occur due to previous choices in the strategy.



- In this example, a strategy for Player 1 must specify an action for the history (C, E), even if it specifies D at the beginning.
- Think of this as allowing for the possibility of mistakes in execution.

Strategy Profiles & Outcomes



- As before, a *strategy profile* is a list of the strategies of all players.
- Given a strategy profile s, the terminal history that results by executing the actions specified by s is denoted $O(s)$, the *outcome* of s.
- For example, in this game, the outcome of the strategy pair (DG, E) is the terminal history D.
- The outcome of (CH, E) is the terminal history (C, E, H).

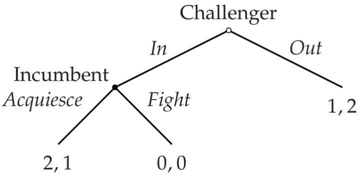
Nash Equilibrium

- **Definition** The strategy profile s^* in an extensive game with perfect information is a **Nash equilibrium** if, for every player i and strategy r_i of player i, the outcome $O(s^*)$ is at least as good as the outcome $O(r_i, s^*_{-i})$ generated by any other strategy profile (r_i, s^*_{-i}) in which player i chooses r_i:

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i$$

- We can construct the *strategic form* of an extensive game by listing all strategies of all players and finding the outcome.

Strategic Form of Entry Game



- The strategic form of the Entry Game is:

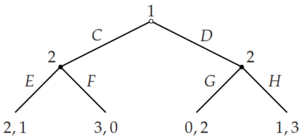
		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

- There are two Nash equilibria: $(In, Acquiesce)$ and $(Out, Fight)$.
- The first NE is the same as the one found with backwards induction.
- In the second NE, the incumbent chooses *Fight*. However, if *In* is taken as given, this is not rational. This is called an *incredible threat*.
- If the incumbent could commit to *Fight* at the beginning of the game, it would be credible.

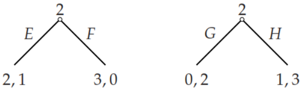
Subgames

- The concept of Nash equilibrium ignores the sequential structure of an extensive game.
- It treats strategies as choices made once and for all at the beginning of the game.
- However, the equilibria of this method may contain incredible threats.
- We'll define a notion of equilibrium that excludes incredible situations.
- Suppose Γ is an extensive form game with perfect information.
- The *subgame* following a non-terminal history h, $\Gamma(h)$, is the game beginning at the point just after h.
- A *proper subgame* is a subgame that is not Γ itself.

Subgames

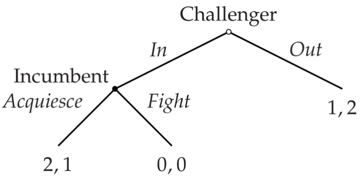


- This game has two proper subgames:



Subgame Perfect Equilibria

- A *subgame perfect equilibrium* is a strategy profile s^* in which each subgame's strategy profile is also a Nash equilibrium.
- Each player's strategy must be optimal for all subgames that have him moving at the beginning, not just the entire game.



- $(Out, Fight)$ is a NE, but is not a subgame perfect equilibrium because in the subgame following *In*, the strategy *Fight* is not optimal for the incumbent.

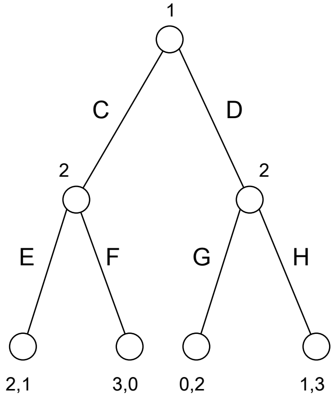
Subgame Perfect Equilibria

- Every subgame perfect equilibrium is also a Nash equilibrium, but not vice versa.
- A subgame perfect equilibrium induces a Nash equilibrium in every subgame.
- In games with finite histories, subgame perfect equilibria are consistent with backwards induction.

Backwards Induction in Finite-Horizon Games

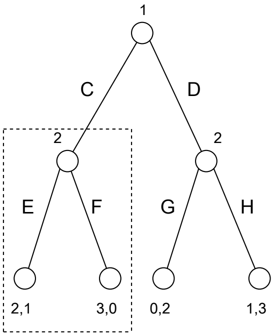
- In a game with a *finite horizon* (i.e. finite maximum length of all terminal histories), we can find all SPNE through backwards induction.
- This procedure can be interpreted as reasoning about how players will behave in future situations.
- Procedure:
  - For all subgames of length 1 (i.e. 1 action away from a terminal node), find the optimal actions of the players.
  - Take these actions as given. For all subgames of length 2, find the optimal actions of the players...
  - Repeat until we cover the entire tree.

Example 1



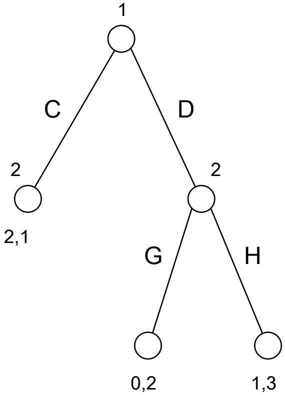
- There are 2 subgames with length 1.

Example 1



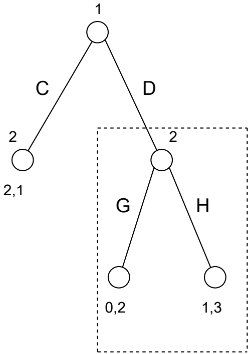
- Consider the left subgame. It is Player 2's turn to move.
- Player 2's optimal action is E, resulting in payoff $(2, 1)$.
- We will assume Player 2 always chooses E, so the payoff of this subgame is $(2, 1)$.

Example 1



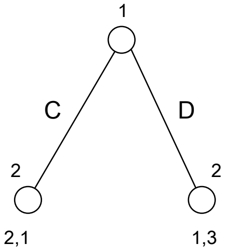
- Therefore, the payoff to Player 1 choosing C is $(2, 1)$.

Example 1



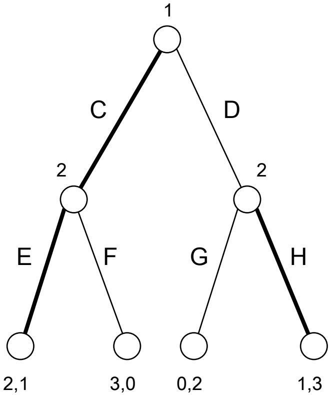
- Consider the right subgame. It is Player 2's turn to move.
- Player 2's optimal action is H, resulting in payoff $(1, 3)$.
- We will assume Player 2 always chooses H, so the payoff of this subgame is $(1, 3)$.

Example 1



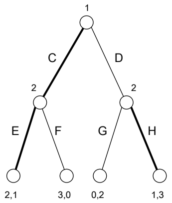
- Therefore, the payoff to Player 1 choosing D is $(1, 3)$.
- Now, Player 1's optimal action is C.
- Backwards induction gives the strategy pair (C, EH).
- The outcome of (C, EH) is the terminal history CE with payoff $(2, 1)$.

Example 1



- We mark the optimal actions at each node with thick lines.

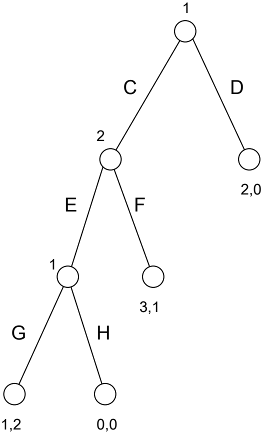
Strategic Form of Example 1



	EG	EH	FG	FH
C	2,1	2,1	3,0	3,0
D	0,2	1,3	0,2	1,3

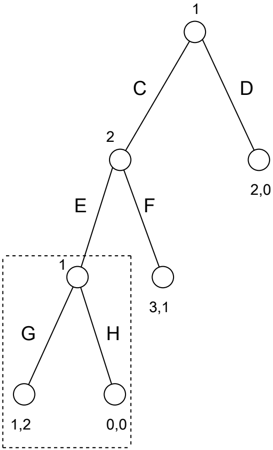
- Let's compare the backwards induction result $(C, EH)$to the NE of the strategic form.
- (C, EG) and (C, EH) are NE of strategic form.
- However, (C, EG) includes a non-optimal action for Player 2 in the right subgame, so is not a subgame-perfect NE.

Example 2



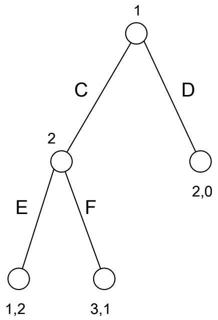
- There is one subgame with length 1, and one subgame with length 2.

Example 2



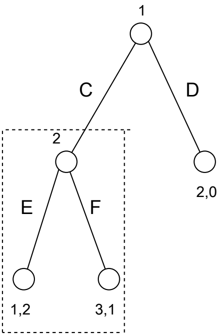
- In this subgame, it is Player 1's turn to move.
- Optimal action is G, resulting in payoff $(1, 2)$.

Example 2



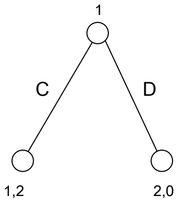
- Assume Player 1 chooses G with certainty.
- Then, the payoff to choosing E is $(1, 2)$.

Example 2



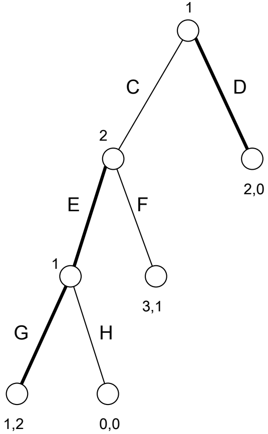
- In this subgame, it is Player 2's turn to move.
- Optimal action is E, resulting in payoff $(1, 2)$.

Example 2



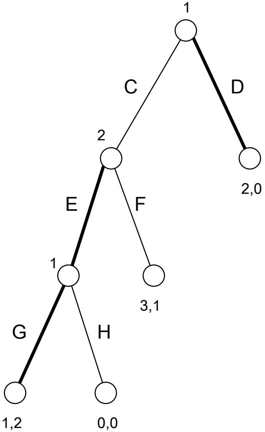
- Optimal action is D, resulting in payoff $(2, 0)$.
- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.

Example 2



- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.

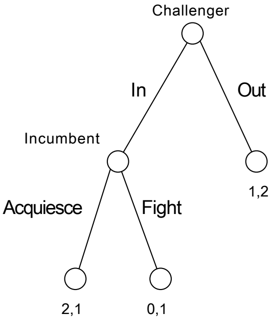
Strategic Form of 2



	E	F
CG	1,2	3,1
CH	0,0	3,1
DG	2,0	2,0
DH	2,0	2,0

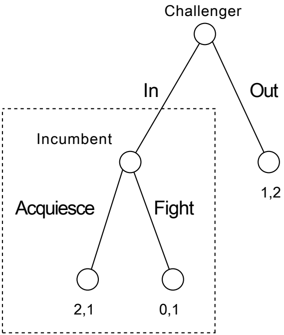
- NE of strategic form are: $(CH, F), (DG, E), (DH, E)$.
- Only (DG, E) is a subgame perfect NE.

Variant of Entry Game



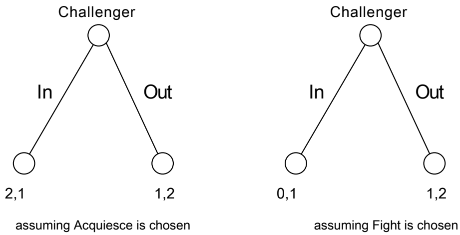
- What if there are multiple optimal actions in a subgame? Then we need to keep track of them separately.
- This is a variant of the entry game in which the Incumbent is indifferent between *Acquiesce*, *Fight*.

Variant of Entry Game



- In this subgame, both *Acquiesce* and *Fight* are optimal actions.
- We cannot eliminate either as an irrational choice. So, we keep track of both possibilities.

Variant of Entry Game

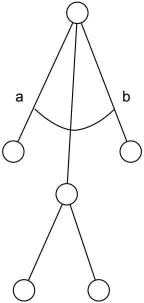


- Backwards induction gives $(In, Acquiesce)$ and $(Out, Fight)$.
- In this case, the NE of the strategic form are the same as the subgame-perfect NE.

Continuous Action Sets

- The action set at a node may be infinite (e.g. if the player chooses a real number).
- In this case, we graphically represent this with an arc between the lowest and highest possible values.
- Effectively, there are an *infinite* number of branches in the game tree at this node.
- Suppose it is Player i's turn to move after all of these branches. Then Player i's strategy profile must specify an action for *all* possible branches.

Continuous Action Sets

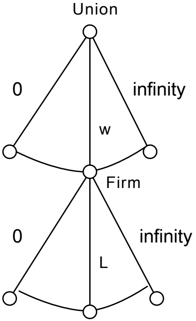


- If the infinite set of actions is an interval of real numbers $[a, b]$, then Player i's strategy profile for this node must be a *function* over $[a, b]$.
- For a strategy profile to be a subgame perfect NE, it must induce a NE at each of the infinite subgames.

Firm-Union Bargaining

- A union and a firm are bargaining.
- First, the union presents a wage demand $w \geq 0$.
- The firm chooses an amount $L \geq 0$ of labor to hire.
- The firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labor, and 2500 if $L > 50$.
- The price of output is 1.
- The firm's preferences are represented by its profits.
- The union's preferences are represented by the total wage bill, wL.

Firm-Union Bargaining



- The firm's payoff is its profit, given by:

$$\Pi(w,L) = \begin{cases} L(100-L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- Union's payoff: wL

Firm-Union Bargaining

$$\Pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- For every $w \geq 0$, there is a subgame where the firm's payoff depends on w.
- Profit has a quadratic part (if $L \leq 50$) and a linear part (if $L > 50$), and is continuous at $L = 50$.
- We want to find the profit-maximizing choice of L.
- The linear part is decreasing in L, so we can ignore it (its maximum is at $L = 50$).
- Quadratic part is maximized at $L^* = \frac{100-w}{2}$.

Firm-Union Bargaining

- Quadratic part is maximized at $L^* = \frac{100-w}{2}$.
- Firm's profit is:

$$\frac{100-w}{2}(100 - \frac{100-w}{2}) - w\frac{100-w}{2} = \frac{(w-100)^2}{4}$$

- Profit is always non-negative. Firm's best response correspondence is:

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

Firm-Union Bargaining

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

- Now, consider the union's decision.
- If $w = 0$ or $w > 100$, union's payoff is 0.
- $wB_f(w) = \frac{w(100-w)}{2}$ is maximized at $w^* = 50$.
- $L^* = B_f(50) = 25$.

Firm-Union Bargaining

- The set of subgame perfect NE is:
- Union's strategy profile: at the empty history, choose $w = 50$.
- Firm's strategy profile: at the subgame following the history w, choose an element of $B_f(w)$.
- Note that the firm has an infinite number of strategy profiles, but there is only one equilibrium outcome, since only the subgame after $w = 50$ will be realized.
- Firm's payoff is 625 and union's payoff is 1250.

Firm-Union Bargaining

- Is there an outcome that both players prefer to the SPNE outcome with payoffs $(1250, 625)$?
- Suppose that instead of each player maximizing his own payoff, a *social planner* could choose both w and L.
- The sum of payoffs is $L(100 - L) - wL + wL = L(100 - L)$ which is maximized at $L = 50$. The choice of w then allocates payoffs to the firm and union.
- For example, if $w = 30$, then the firm's payoff is 1000 and the union's payoff is 1500.
- This is an illustration that individual maximization may not achieve the most efficient outcome.

Firm-Union Bargaining

- Is there a Nash equilibrium outcome that differs from any subgame perfect NE outcome?
- Suppose the union's strategy is: offer $w = 100$ and the firm's strategy profile is: for any w, offer $L = 0$.
- The firm has no incentive to deviate, since it will make a negative payoff for any $L > 0$.
- The union has no incentive to deviate, because it will get a payoff of 0 for any choice of w.
- This is not subgame perfect, since the firm's strategy is not optimal for $w < 100$.

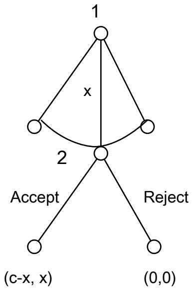
Characteristics of Perfect-Information, Finite Horizon Games

- **Theorem 7.4**: Every backwards induction strategy for a perfect-information, finite, extensive form game is also a NE.
- **Corollary 7.1**: Every finite extensive game with perfect information has a pure strategy NE.

Example: Ultimatum Game

- This is a simple game that can model a very simplified bargaining situation.
- Two people want to split some amount $c > 0$. The procedure is as follows:
  - First, Player 1 offers $0 \geq x \geq c$ to Player 2.
  - Then, Player 2 chooses to *Accept* or *Reject* the offer.
  - If he chooses *Accept*, payoffs are: $c - x$ for Player 1 and x for Player 2.
  - If he chooses *Reject*, both players get 0.

Example: Ultimatum Game



- Suppose $c = 5$. I will choose three people to be Player 1, and three people to be Player 2.
- Player 1 will write down an offer between 0 and 5.
- Player 2 will write down *Accept* or *Reject*.

SPNE of Ultimatum Game

- We can use backwards induction to find the SPNE of this game.
- Consider the last subgame. Taking x as given, Player 2 will optimally *Accept* if $x > 0$, and may choose either *Accept* or *Reject* if $x = 0$.
- Two possible strategies for Player 2:
  - Player 2 *Accepts* all offers $x \geq 0$, and *Rejects* all other offers
  - Player 2 *Accepts* all offers $x > 0$, and *Rejects* all other offers

SPNE of Ultimatum Game

- Player 1's decision: consider each of Player 2's possible strategies separately.
- If Player 2 *Accepts* all offers $x \geq 0$:
  - Player 1's optimal offer is $x = 0$.
  - Player 2 will *Accept*, leading to payoffs $(c, 0)$.
- If Player 2 only *Accepts* offers $x > 0$:
  - There is no optimal offer for Player 1: it is better to offer x as close as possible to 0 while still being > 0.
  - Therefore, the second strategy cannot be part of a SPNE.
- We are left with a single SPNE: Player 1 offers $x = 0$, and Player 2 accepts all offers $x \geq 0$.

Experimental Testing of the Ultimatum Game

- The SPNE solution concept makes a very clear prediction: Player 1 offers zero, and Player 2 accepts all offers $x \geq 0$.
- However, when people play the Ultimatum Game in experiments, they consistently choose a different result.
- In experiments, Player 1 offers around $0.3c$. Player 2 chooses *Reject* about 20% of the time.
- Why the difference? Two possible explanations:
  - Equity: real people also value equity or "fairness", but the players in the model only care about their own payoff.
  - Repeated interactions: in real life, people interact repeatedly, so Player 2 can choose *Reject* to develop a reputation for punishing low offers. In the game, there is only one interaction.

Example: Fair Division of a Cake

- Suppose two people want to divide a cake into two pieces such that both people will be satisfied, *without* asking a third person to divide it.
- Suppose the total cake has size=1.
  - Player 1 cuts the cake into two pieces (chooses a number x between 0 and 1);
  - Player 2 chooses the piece he prefers (chooses either x or $1 - x$).
  - Each player's payoff is equal to the size of the piece they receive.

Example: Fair Division of a Cake

- Player 2's decision: suppose x has been chosen by Player 1.
- If $x < \frac{1}{2}$, best response is $1 - x$.
- If $x > \frac{1}{2}$, best response is x.
- If $x = \frac{1}{2}$, either is a best response (let's suppose that Player 2 chooses x).
- Assuming this strategy of Player 2, then Player 1's payoff as a function of x is:
  - If $x < \frac{1}{2}$, payoff will be x.
  - If $x \geq \frac{1}{2}$, payoff will be $1 - x$.
- Player 1's best response is $x = \frac{1}{2}$.

Administrative Stuff

- The final exam will be on June 23 from 13:30-15:30 in Boxue 504. It will cover the material after the midterm exam.
- HW #4 is due today at the end of class. I will post the solutions and the last homework on the class web site.
- There will be no class next Monday, June 5.
- The last class will be June 12.