Advanced Microeconomic Analysis, Lecture 2

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Homework #1

- I will post Homework #1 on the course website later tonight.
- http://rncarpio.com/teaching/AdvMicro
- Due in two weeks at the end of lecture.
- This week, we will begin Chapter 1 of Jehle & Reny.

Review of Last Lecture

- A set S is *convex* if any convex combination of points in S, also lies in S.
- A function f is *concave* if:
  - Its hypograph (i.e. the set of points below the graph) is convex.v
  - Assuming twice differentiable, its Hessian (i.e. matrix of second derivatives) is negative semidefinite.
- A function f is *quasiconcave* if its upper level sets, i.e. $\{\mathbf{x} | f(x) \geq r\}$ are convex, for all r.
  - Example: a "bell curve" or normal distribution is quasiconcave, but not concave.

Review of Last Lecture: Constrained Optimization

- For problems with one *equality* constraint $g(\boldsymbol{x}) = 0$, we can form the Lagrangian and find its inflection point:

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda g(\boldsymbol{x})$$

- Set $\frac{\partial L}{\partial \boldsymbol{x}}, \frac{\partial L}{\partial \lambda}$ to 0 and solve for $\boldsymbol{x}^*, \lambda^*$.
- At the solution, the multiplier λ^* must be strictly positive.

Review of Last Lecture: Constrained Optimization

- For problems with multiple *inequality* constraints $g^1(\boldsymbol{x}) \le 0, ..., g^m(\boldsymbol{x}) \le 0$, the Lagrangian becomes:

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda_1 g^1(\boldsymbol{x}) ... - \lambda_m g^m(\boldsymbol{x})$$

- At the solution, some constraints will be *binding* (i.e. satisfied with equality) or *nonbinding* (satisfied with strict inequality).
- Of the binding constraints, some may be *irrelevant* (i.e. if the constraint were removed, the solution would not change).
- The multipliers λ_i^* corresponding to nonbinding and irrelevant constraints will be 0.
- The multipliers λ_i^* corresponding to binding and relevant constraints will be strictly positive.

Consumer Theory

- The consumer's problem is to choose a consumption bundle, out of all feasible bundles, that is most preferred.
- There are four building blocks of the consumer's problem:
    - the *consumption set*
    - the *feasible set*
    - the *preference relation*
    - the *behavioral assumption*

Consumption & Feasible Set

- The consumption set X is the set of all possible consumption bundles.
- We will assume it is $\mathbb{R}^n_+$, the set of all possible non-negative quantities of n goods.
- The feasible set B is a subset of X, representing the set of consumption bundles achievable with the consumer's resources (i.e. wealth).

Preferences

- The consumer's preferences determine which bundles are more preferred.
- We will characterize preferences *axiomatically*: we will state the minimum assumptions which any reasonable consumer preference must satisfy.
- Let $\mathbf{x^1}, \mathbf{x^2}$ be any two elements of the consumption set X. We say $\mathbf{x^1} \succsim \mathbf{x^2}$ if $\mathbf{x^1}$ is at least as good as $\mathbf{x^2}$.

Axioms of Consumer Choice

- Axiom 1: Completeness. For all $\mathbf{x^1}, \mathbf{x^2} \in X$, either $\mathbf{x^1} \succsim \mathbf{x^2}$ or $\mathbf{x^2} \succsim \mathbf{x^1}$.
- Axiom 1 implies that there are no noncomparabilities, i.e. bundles that cannot be compared to each other.
- Axiom 2: Transitivity: for any three elements $\mathbf{x^1}, \mathbf{x^2}, \mathbf{x^3}$, if $\mathbf{x^1} \succsim \mathbf{x^2}$ and $\mathbf{x^2} \succsim \mathbf{x^3}$, then $\mathbf{x^1} \succsim \mathbf{x^3}$.
- Axiom 2 implies that preferences are *consistent*. If preferences were not consistent, it would be possible to extract an infinite amount of money from a consumer though a *money pump*.

Money Pump

- Suppose a consumer has the following preferences: $\boldsymbol{x}^1 \succsim \boldsymbol{x}^2$, $\boldsymbol{x}^2 \succsim \boldsymbol{x}^3$, $\boldsymbol{x}^3 \succsim \boldsymbol{x}^1$.
- There is a "cycle" of preferences.
- Suppose the consumer starts out with $\boldsymbol{x}^1$.
- I trade $\boldsymbol{x}^3$ to the consumer for $\boldsymbol{x}^1$ plus a small amount of money.
- Then, I trade $\boldsymbol{x}^2$ to the consumer for $\boldsymbol{x}^3$ plus a small amount of money.
- Finally, I trade $\boldsymbol{x}^1$ to the consumer for $\boldsymbol{x}^2$ plus a small amount of money.
- This can be repeated indefinitely.

Strict Preference

- A binary relation $\succsim$ is called a *preference relation* if it satisfies the axioms of completeness and transitivity.
- We can define another binary relation, $\succ$, defined as:

$$\mathbf{x^1} \succ \mathbf{x^2} \text{ if and only if } \mathbf{x^1} \succsim \mathbf{x^2} \text{ and } \mathbf{x^2} \not\succsim \mathbf{x^1}$$

- This is called the *strict preference relation* induced by $\succsim$.
- If $\mathbf{x^1} \succ \mathbf{x^2}$, we say $\mathbf{x^1}$ is *strictly preferred* to $\mathbf{x^2}$.

Indifference Relation

- The binary relation $\sim$ is defined as:

$$\mathbf{x^1} \sim \mathbf{x^2} \, iff \, \mathbf{x^1} \succsim \mathbf{x^2} \text{ and } \mathbf{x^2} \succsim \mathbf{x^1}$$

- We say that $\mathbf{x^1}$ is *indifferent* to $\mathbf{x^2}$.

Sets Derived from $\succsim$

- Given a preference relation $\succsim$, we can define sets relative to a given point $\mathbf{x^0}$.
- Define the set

$$\succsim(\mathbf{x^0}) = \{\mathbf{x} | \mathbf{x} \succsim \mathbf{x^0}\}$$

- This is the set of all bundles that are *at least as good as* $\mathbf{x^0}$.
- Similarly, we can define $\precsim(\mathbf{x^0}), \prec(\mathbf{x^0}), \succ(\mathbf{x^0}), \sim(\mathbf{x^0})$.

Continuity, Local Non-satiation

- Axiom 3: Continuity. For all $\boldsymbol{x} \in \mathbb{R}^n_+, \succsim(\boldsymbol{x})$ and $\precsim(\boldsymbol{x})$ are *closed* (i.e. contains its boundary).
- The next two axioms are alternatives:
- Axiom 4': Local non-satiation. For all $\boldsymbol{x^0} \in \mathbb{R}^n_+$ and for all $\epsilon > 0$, there exists some $\boldsymbol{x} \in B_\epsilon(\boldsymbol{x^0})$ such that $\boldsymbol{x} \succ \boldsymbol{x^0}$.
  - $B_\epsilon(\boldsymbol{x^0})$ is a ball of radius ϵ centered on $\boldsymbol{x^0}$.
  - This axiom says that from any point $\boldsymbol{x^0}$, you can always find a path that leads to a strictly more preferred bundle.
  - This rules out local maxima.

Strict Monotonicity

- Axiom 4: Strict monotonicity. For all $\mathbf{x^0}, \mathbf{x^1}$:
  - If $\mathbf{x^0} \geq \mathbf{x^1}$ (i.e. every component of $\mathbf{x^0}$ is at least as large as in $\mathbf{x^1}$), then $\mathbf{x^0} \succsim \mathbf{x^1}$.
  - If $\mathbf{x^0} \gg \mathbf{x^1}$ (i.e. every component of $\mathbf{x^0}$ is at least as large as in $\mathbf{x^1}$, *and* there is one strictly greater), then $\mathbf{x^0} \succ \mathbf{x^1}$.
- This is a stricter condition that local non-satiation: it says that from any point $\mathbf{x^0}$, any path in a direction that increases a component of $\mathbf{x}$ leads to a strictly more preferred bundle.
- In the consumer problem, Axioms 4 and 4' guarantee that the chosen bundle will lie on the budget line (therefore, we can use the Lagrange method with an equality condition).

Convexity and Strict Convexity

- Axiom 5': Convexity. If $\mathbf{x^1} \succsim \mathbf{x^0}$, then $t\mathbf{x^1} + (1-t)\mathbf{x^0} \succsim \mathbf{x^0}$ for all $t \in [0,1]$.
- Axiom 5: Strict Convexity: If $\mathbf{x^1} \neq \mathbf{x^0}$ and $\mathbf{x^1} \succsim \mathbf{x^0}$, then $t\mathbf{x^1} + (1-t)\mathbf{x^0} \succ \mathbf{x^0}$ for all $t \in [0,1]$.
- Axiom 5' implies that the set $\succsim (\mathbf{x^0})$ is convex for any $x0$. Axiom 5 implies that it is *strictly* convex.
- We shall see later that this implies the utility function is (strictly) quasiconcave if (strict) convexity is satisfied.
- For the rest of this course, unless specified, we will usually assume Axiom 4 (strict monotonicity) and 5 (strict convexity).

The Utility Function

- We can summarize preferences with a *utility function* $u(\cdot)$, that assigns a number fo every consumption bundle $\boldsymbol{x}$.
- $u(\boldsymbol{x}^0) \geq u(\boldsymbol{x}^{\mathbf{1}})$ iff $\boldsymbol{x}^{\mathbf{0}} \succsim \boldsymbol{x}^{\mathbf{1}}$.
- $u(\boldsymbol{x}^0) > u(\boldsymbol{x}^{\mathbf{1}})$ iff $\boldsymbol{x}^{\mathbf{0}} \succ \boldsymbol{x}^{\mathbf{1}}$.
- **Theorem 1.1**: If $\succsim$ is complete, transitive, continuous and strictly monotonic, there exists a continuous real-valued function, $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succsim$.

Invariance to Positive Monotonic Transforms

- The utility function $u(\cdot)$ representing $\succsim$ is not unique; there are infinitely many possibilities.
- Suppose we have a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, and let $v(\boldsymbol{x}) = f(u(\boldsymbol{x}))$. Then $v(\cdot)$ is also a utility function that represents $\succsim$, and vice versa.
- For example: $u_1(x) = x$, a linear function, and $u_2(x) = \log(x)$, a log function, represent the same preferences.
- For any two x, y, $u_1(x) \leq u_1(y)$ if and only if $u_2(x) \leq u_2(y)$.

Properties of Preferences and Utility Functions

- Suppose $\succsim$ is represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:
- 1) $u(\boldsymbol{x})$ is strictly increasing iff $\succsim$ is strictly monotonic.
- 2) $u(\boldsymbol{x})$ is quasiconcave iff $\succsim$ is convex.
- 3) $u(\boldsymbol{x})$ is strictly quasiconcave iff $\succsim$ is strictly convex.
- As we will see, strict quasiconcavity guarantees that the consumer problem has a *unique* solution (most preferred bundle).

Consumer's problem

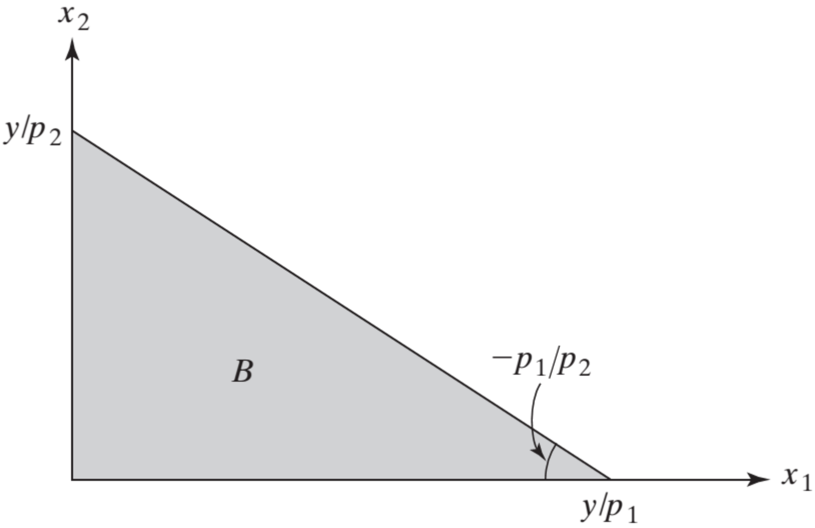
- Consumer's problem is to choose a bundle $\mathbf{x} = (x_1 \ldots x_n)$ that is *most preferred*, from the feasible set B
- Choose:

$$\mathbf{x}^* \in B \text{ such that } \mathbf{x}^* \succsim \mathbf{x}, \text{ for all } \mathbf{x} \in B$$

- Assume the consumer is in a *market economy*, in which all transactions occur in markets
- For all goods $i = 1 \ldots n$, there is a *market price* $p_i > 0$
- Assume the consumer has no *market power*, i.e. cannot affect market price through buying and selling
- Assume the consumer has an exogenously given amount of money, y.
- Total expenditures, $\sum_{i=1}^n p_i x_i$ must be less than or equal to y.
- Then, given prices $\mathbf{p} = (p_1 \ldots p_n)$, the feasible set (or *budget set*) is:

$$B = \{\mathbf{x} \in \mathbb{R}^n_+ | \mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^n p_i x_i \leq y\}$$

Budget set for 2 goods



Consumer's problem with utility functions

- Assume that preferences can be represented by a utility function $u(\boldsymbol{x})$, and that $u(\boldsymbol{x})$ is strictly quasiconcave.
- Consumer's problem becomes:

$$\max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}) \text{ subject to } \boldsymbol{p} \cdot \boldsymbol{x} \leq y$$

- What do we know about the solution $\boldsymbol{x}^*$?
  - A maximum *exists*, because B is closed and bounded.
  - The maximizer $\boldsymbol{x}^*$ is *unique*, because $u(\boldsymbol{x})$ is strictly quasiconcave.
  - $\boldsymbol{x}^*$ satisfies the budget constraint *with equality*, i.e. $\boldsymbol{p} \cdot \boldsymbol{x} = y$, because $u(\boldsymbol{x})$ is strictly monotonic.
- It would be convenient if we could guarantee all components x_i^* of $\boldsymbol{x^*}$ were strictly positive (i.e. an interior solution).
- Choosing a specific form for $u(\boldsymbol{x})$ can ensure this.
- The solution $x_i^*(p_1, ...p_n, y)$, as a function of $\boldsymbol{p}$ and y, is called the *Marshallian demand function*.

Solving the consumer's problem with calculus methods

- If we make the further assumption that $u(\boldsymbol{x})$ is *differentiable* (i.e. smooth), we can use the Lagrangian method.
- The constraint $g(x_1, ...x_n) = 0$ becomes: $p_1x_1 + ... + p_nx_n - y = 0$

$$L(x_1...x_n, \lambda) = u(x_1, ...x_n) - \lambda(p_1x_1 + ... + p_nx_n - y)$$

- Assume that the solution $\boldsymbol{x}^*$ is strictly positive. Then the first-order conditions (called the *Kuhn-Tucker* conditions) are satisfied with some $\lambda^* > 0$:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(x^*)}{\partial x_1} - \lambda^* p_1 = 0$$
$$\vdots$$
$$\frac{\partial L}{\partial x_n} = \frac{\partial u(x^*)}{\partial x_n} - \lambda^* p_n = 0$$
$$p_1x_1 + ... + p_nx_n - y = 0$$

Marginal rate of substitution = price ratio

- Rearrange first-order conditions:

$$\frac{\partial u(x^*)}{\partial x_1} = \lambda^* p_1$$

$$\vdots$$

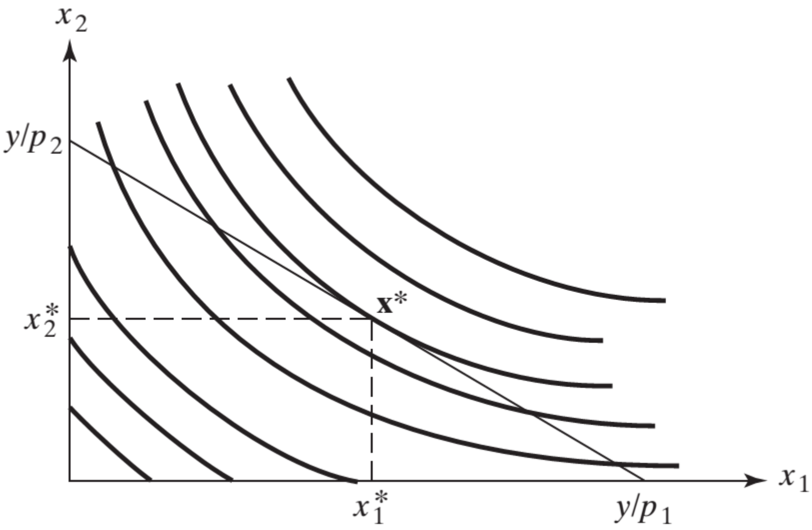
$$\frac{\partial u(x^*)}{\partial x_n} = \lambda^* p_n$$

- Assume that $\frac{\partial u(x^*)}{\partial x_1} > 0$, i.e. utility is always increasing in each good. Then for any two goods j, k:

$$\frac{\frac{\partial u(\mathbf{x^*})}{\partial x_j}}{\frac{\partial u(\mathbf{x^*})}{\partial x_j}} = \frac{p_j}{p_k}$$

- At the optimum, the marginal rate of substitution between any two goods equals the ratio of prices.

Solution for 2 goods



Marshallian demand function: 2 goods

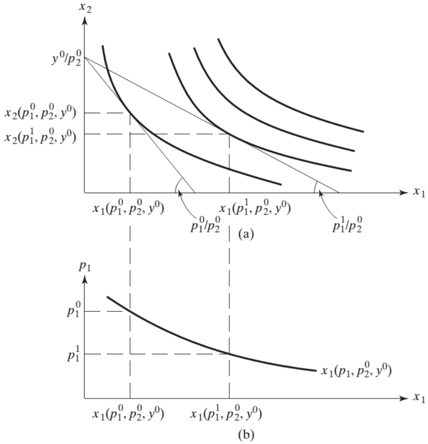


Figure 1.11. The consumer's problem and consumer demand behaviour.

Example: Constant Elasticity of Substitution (CES) utility

- For $n = 2$ and a constant $\rho < 1, \rho \neq 0$, assume utility is of the form:

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

- Is this strictly monotonic? Check that the gradient is always strictly positive for $x_1, x_2 \geq 0$:

$$\frac{\partial u}{\partial x_1} = ((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1}$$

$$\frac{\partial u}{\partial x_2} = ((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1}$$

- Note that this goes to infinity as $x_i \to 0$. This ensures that a maximizer will not have $x_1 = 0$ or $x_2 = 0$.
- Is this strictly quasiconcave? We'll mention 2 ways of checking:

Hessian of the Utility Function

- The Hessian of $u(x_1, x_2)$ is:

$$H(x_1, x_2) = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} \end{pmatrix}$$

- If $H(x_1, x_2)$ is negative definite for all $x_1 \geq 0, x_2 \geq 0$, then $u(x_1, x_2)$ is strictly concave.
- $H(x_1, x_2)$ is negative semidefinite for all $x_1 \geq 0, x_2 \geq 0$, iff $u(x_1, x_2)$ is concave.
- A matrix is negative definite if the determinants of its *upper left* $1x1, 2x2, ...$ sub-matrices are negative.
- To determine if u is quasiconcave, we construct the *bordered Hessian*.
- This method works on any function. However, it's a bit tedious to calculate the determinants.

Composition of Concave Functions

- We can use these properties of concave and quasiconcave functions:
- A (strictly) concave, monotonic transformation of a (strictly) concave function is (strictly) concave.
- A monotonic transformation of a concave function is quasiconcave (not necessarily concave).
- A monotonic transformation of a quasiconcave function is quasiconcave.
- For example: a negative quadratic function $-\frac{1}{2}x^2$ is concave.
- Applying the monotonic transformation $\exp(\cdot)$ to $-\frac{1}{2}x^2$ gives $\exp(-\frac{1}{2}x^2)$, which is a "bell curve" or normal distribution, times a constant.

Composition of Concave Functions

- For the CES utility $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$:
- $0 < \rho < 1$, let $f(x_1, x_2) = x_1^\rho + x_2^\rho$, and let $g(z) = z^{\frac{1}{\rho}}$.
- For $\rho < 0$, let $f(x_1, x_2) = -(x_1^\rho + x_2^\rho)$, and let $g(z) = (-z)^{\frac{1}{\rho}}$.
- g is strictly concave and monotonic, and f is strictly concave, therefore $u = g \circ f$ is strictly concave.

CES Example: Consumer's Problem

$$\max_{x_1, x_2} (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} \text{ subject to } p_1 x_1 + p_2 x_2 - y \leq 0$$

.

- Since u is strictly monotonic, the budget constraint will hold with equality at the solution: $p_1 x_1 + p_2 x_2 - y = 0$.
- The Lagrangian function:

$$L(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

- As we saw above, the solution is strictly positive, so the Kuhn-Tucker conditions hold:

$$\frac{\partial L}{\partial x_1} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - y = 0$$

- Using the property that MRS = price ratio:

$$\frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{p_1}{p_2} \Rightarrow \frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

- Plugging into budget equation,

$$y = p_1 x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2 x_2$$

- Rearranging gives us:

$$x_2 = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} y, x_1 = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} y$$

- These are the solutions to the consumer's problem for a given value of p_1, p_2, y, i.e. the Marshallian demand function.

Marshallian demand function: CES

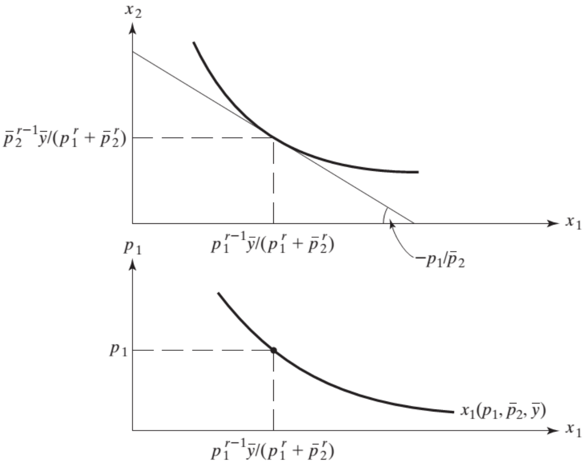


Figure 1.12. Consumer demand when preferences are represented by a CES utility function.

Conditions for a Differentiable Demand Function

- Assume that $\boldsymbol{x}^*$ solves the consumer's maximization problem given $\boldsymbol{p}^0, y^0$ and is strictly positive.
- If
  - u is twice continuously differentiable on $\mathbb{R}^n_{++}$ (i.e. $\boldsymbol{x}$ strictly positive)
  - $\frac{\partial u}{\partial x_i} > 0$ for some i in $1...n$
  - the bordered Hessian of u has a non-zero determinant at $\boldsymbol{x}^*$,
- then $\boldsymbol{x}(\boldsymbol{p}, y)$ is differentiable at $(\boldsymbol{p}^0, y^0)$.

Envelope Theorem (Appendix A2.4)

- Let's go back to the general optimization problem

$$\max_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{a}) \text{ subject to } g(\boldsymbol{x},\boldsymbol{a}) = 0$$

- $\boldsymbol{x}$ is the vector of choice variables, and $\boldsymbol{a}$ is a vector of parameters (for consumer problems, prices and income).
- Suppose that for each $\boldsymbol{a}$, a solution exists: $\boldsymbol{x}^*(\boldsymbol{a})$ and $\lambda^*(\boldsymbol{a})$. The maximized value of the objective function is called the *value function*:

$$V(\boldsymbol{a}) = f(\boldsymbol{x}^*(\boldsymbol{a}),\boldsymbol{a}) = \max_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{a}) \text{ s.t. } g(\boldsymbol{x},\boldsymbol{a}) = 0$$

Envelope Theorem (Appendix A2.4)

- We would like to know how changes in $\boldsymbol{a}$ affects the *optimized objective function* $V(\boldsymbol{a})$.
- To simplify, the Envelope Theorem (A2.22 in the book) states that the effect on V is equal to the partial derivative of the Lagrangian, evaluated at the optimal values of $\boldsymbol{x}^*, \lambda^*$:

$$\frac{\partial V}{\partial a_j} = \frac{\partial L}{\partial a_j}(\boldsymbol{x}^*(\boldsymbol{a}), \lambda^*(\boldsymbol{a}))$$

Indirect Utility Function

- Going back to the consumer's problem, let $v(\boldsymbol{p}, y)$ denote the value function, i.e. the maximized utility function:

$$v(\boldsymbol{p}, y) = u(\boldsymbol{x}^*(\boldsymbol{p}, y)) = \max_{\boldsymbol{x}} u(\boldsymbol{x}) \text{ s.t. } \boldsymbol{p} \cdot \boldsymbol{x} \leq y$$

- $u(\boldsymbol{x})$ is called the *direct utility function*, while $v(\boldsymbol{p}, y)$ is called the *indirect utility function*.
- $v(\boldsymbol{p}, y)$ gives the highest amount of utility the consumer can reach, given prices $\boldsymbol{p}$ and income y.

Indirect Utility

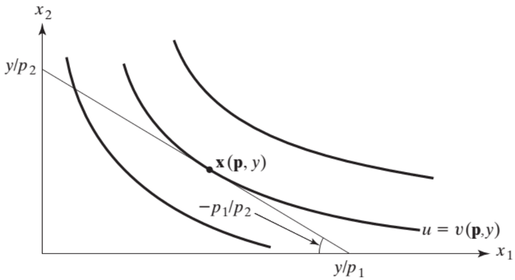


Figure 1.13. Indirect utility at prices **p** and income *y*.

Properties of Indirect Utility

- If $u(\boldsymbol{x})$ is continuous and strictly increasing, then $v(\boldsymbol{p}, y)$ is:
  - Continuous
  - Homogeneous of degree zero in $(\boldsymbol{p}, y)$: for any $t > 0, v(t\boldsymbol{p}, ty) = v(\boldsymbol{p}, y)$
  - Strictly increasing in y
  - Decreasing in $\boldsymbol{p}$.
  - Quasiconvex in $(\boldsymbol{p}, y)$
  - Satisfies *Roy's Identity*: if $v(\boldsymbol{p}, y)$ is differentiable at $(\boldsymbol{p}^0, y^0)$ and $\frac{\partial v}{\partial y}(\boldsymbol{p}^0, y^0) \neq 0$, then:

$$x_i(\boldsymbol{p}^0, y^0) = -\frac{\frac{\partial v}{\partial p_i}(\boldsymbol{p}^0, y^0)}{\frac{\partial v}{\partial y}(\boldsymbol{p}^0, y^0)} \text{ for } i = 1 \ldots n$$

Homogeneity of Indirect Utility

- A function $f(\boldsymbol{x})$ is *homogeneous of degree p* if for any $t > 0, f(t\boldsymbol{x}) = t^p f(\boldsymbol{x})$.
- For $v(\boldsymbol{p}, y)$, homogeneity of degree 0 means that if both prices and income are multiplied by the same amount, the optimal consumption bundle does not change, and therefore the maximized utility does not change.
- $\boldsymbol{x}(\boldsymbol{p}, y)$ is also homogeneous of degree 0.
- There is no "money illusion": changing nominal prices and incomes does not affect the outcome.

CES Example

- Going back to the CES utility example, we found the Marshallian demand functions:

$$x_1 = \frac{p_1^{r-1}}{p_1^r + p_2^r} y$$

$$x_2 = \frac{p_2^{r-1}}{p_1^r + p_2^r} y$$

- where $r = \frac{\rho}{\rho - 1}$. Note that homogeneity of degree zero holds.
- Plugging back into the direct utility function u:

$$v(p_1, p_2, y) = \left[\left(\frac{p_1^{r-1}}{p_1^r + p_2^r} y \right)^\rho + \left(\frac{p_2^{r-1}}{p_1^r + p_2^r} y \right)^\rho \right]^{\frac{1}{\rho}}$$

$$= y \left(p_1^r + p_2^r \right)^{\frac{-1}{r}}$$

- This is also homogeneous of degree zero.
- We can also verify the other properties of an indirect utility function.

Expenditure Function

- The *expenditure function* is the minimum amount of expenditure necessary to achieve a given utility level u at prices $\boldsymbol{p}$:

$$e(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

- If preferences are strictly monotonic, then the constraint will be satisfied with equality
- Denote the solution to the expenditure minimization problem as:

$$\boldsymbol{x}^h(\boldsymbol{p}, u) = \arg\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

- This is called the *Hicksian demand function* or *compensated demand*.
- It shows the effect of a change in prices on demand, while *holding utility constant*.

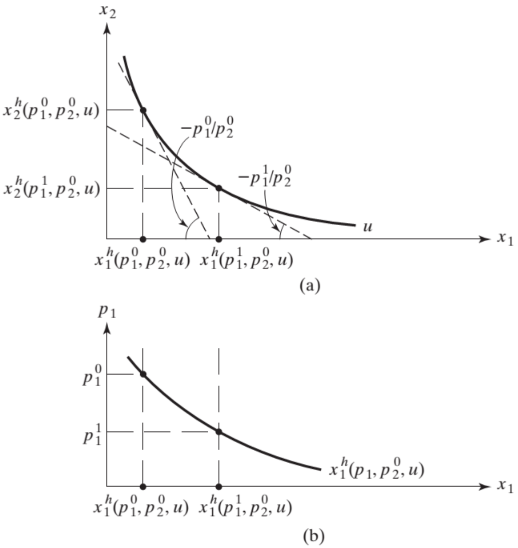


Figure 1.16. The Hicksian demand for good 1.

Properties of Expenditure Function

- If $u(\cdot)$ is continuous and strictly increasing, then $e(\boldsymbol{p}, u)$ is:
  - Zero when u is at the lowest possible level
  - Continuous
  - For all strictly positive $\boldsymbol{p}$, it is strictly increasing and unbounded above in u
  - Increasing in $\boldsymbol{p}$
  - Homogeneous of degree 1 in $\boldsymbol{p}$
  - Concave in $\boldsymbol{p}$
  - If $u(\cdot)$ is strictly quasiconcave, then it satisfies *Shephard's lemma*:

$$\frac{\partial e(\boldsymbol{p}^0, u^0)}{\partial p_i} = x_i^h(\boldsymbol{p}^0, u^0) \qquad \text{for } i = 1 \ldots n$$

Example: CES Utility

- Suppose direct utility is $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$.
- Let's derive the expenditure function:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \qquad \text{s.t.} \qquad (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- Form the Lagrangian:

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u)$$

- First-order conditions:

$$\frac{\partial L}{\partial x_1} = p1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

Example: CES Utility

$$\frac{\partial L}{\partial x_1} = p1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- Solving for x_1, x_2, we get the Hicksian demands ($r = \frac{\rho}{\rho-1}$):

$$x_1^h(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}$$

$$x_2^h(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$$

- Plug back into objective function $\boldsymbol{p} \cdot \boldsymbol{x}$:

$$e(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}$$

Next Week

- I will post Homework #1 on the course website later tonight.
- http://rncarpio.com/teaching/AdvMicro
- Due in two weeks at the end of lecture.
- Please continue reading Chapter 1.