Advanced Microeconomic Analysis, Lecture 3

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Homework #1

- Homework #1 is due next week.
- For next week, please read Chapter 2.1 (Duality: A Closer Look) and continue to Chapter 3. We will not cover the other parts of Chapter 2.

Review of Last Lecture

- The *consumer problem* is to solve

$$\max_{\boldsymbol{x}} u(\boldsymbol{x}) \qquad \text{subject to } \boldsymbol{p} \cdot \boldsymbol{x} \leq y$$

- The maximizer to this problem (assuming it exists and is single-valued), $\boldsymbol{x}^*(\boldsymbol{p}, y)$, is the *Marshallian demand function*.
- The *indirect utility function*, or *value function*, is the maximized value of $u(\boldsymbol{x})$ subject to prices $\boldsymbol{p}$ and income y:

$$v(\boldsymbol{p}, y) = max_{\boldsymbol{x}} u(\boldsymbol{x}) \qquad \text{s.t.} \qquad \boldsymbol{p} \cdot \boldsymbol{x} \leq y$$

- $v(\boldsymbol{p}, y) = u(\boldsymbol{x}^*(\boldsymbol{p}, y))$

Review of Last Lecture

- Properties of indirect utility:
  - Continuous
  - Homogeneous of degree zero in $(\boldsymbol{p}, y)$
  - Strictly increasing in y
  - Decreasing in $\boldsymbol{p}$
  - Quasiconvex in $(\boldsymbol{p}, y)$
  - Roy's identity:

$$x_i(\boldsymbol{p}^0, y^0) = -\frac{\frac{\partial v}{\partial p_i}(\boldsymbol{p}^0, y^0)}{\frac{\partial v}{\partial y}(\boldsymbol{p}^0, y^0)} \text{ for } i = 1 \ldots n$$

Example: Cobb-Douglas utility

- Consider the utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.
- This is a very common utility function in economics, called Cobb-Douglas utility.
- Let's find the Marshallian demand function $\boldsymbol{x}(p_1, p_2, y)$ and indirect utility function $v(p_1, p_2, y)$.

- The consumer problem is:

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{1-\alpha} \qquad \text{subject to} \qquad p_1 x_1 + p_2 x_2 - y = 0$$

- Form the Lagrangian:

$$L(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

- First-order conditions:

$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - y = 0$$

- The MRS = price ratio condition:

$$\frac{\alpha x_2}{(1-\alpha)x_1} = \frac{p_1}{p_2} \qquad \Rightarrow \qquad p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$$

- Plug into the budget equation $p_1 x_1 + p_2 x_2 = y$ to get:

$$\frac{\alpha}{1-\alpha} p_2 x_2 + p_2 x_2 = y$$
$$\left(\frac{\alpha}{1-\alpha} + 1\right) p_2 x_2 = \frac{1}{1-\alpha} p_2 x_2 = y$$
$$x_2^* = (1-\alpha)\frac{y}{p_2}, x_1^* = \alpha \frac{y}{p_1}$$

- Note that $p_1 x_1 = \alpha y$ and $p_2 x_2 = (1-\alpha)y$.
- The exponent of each good, α and $1-\alpha$, determine the fraction of income allocated to each good.

- The indirect utility function is:

$$\begin{aligned} v(p_1, p_2, y) &= u(x_1^*, x_2^*) \\ &= \left(\alpha \frac{y}{p_1}\right)^{\alpha} \left((1-\alpha)\frac{y}{p_2}\right)^{1-\alpha} \\ &= \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} y \end{aligned}$$

- We can verify that this is homogeneous of degree zero in p_1, p_2, y.
- Let's check Roy's identity:

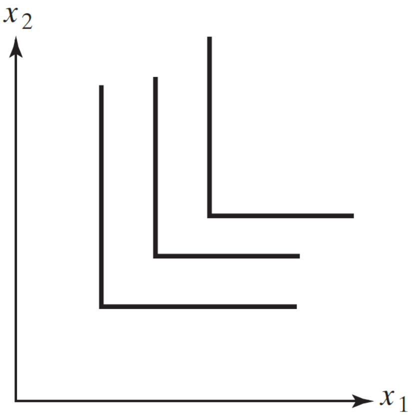
$$\begin{aligned} -\frac{\frac{\partial v}{\partial p_i}(\boldsymbol{p}^0, y^0)}{\frac{\partial v}{\partial y}(\boldsymbol{p}^0, y^0)} &= -\frac{-\alpha\alpha^{\alpha} p_1^{-\alpha-1} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} y}{\left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}} \\ &= \alpha \frac{y}{p_1} \end{aligned}$$

A Non-Differentiable Utility Function

- Consider the utility function $u(x_1, x_2) = \min(x_1, x_2)$. This is called *Leontief utility*.
- It is non-differentiable, so we cannot use the Lagrangian method to solve the utility maximization problem.
- If $x_1 \le x_2, u(x_1, x_2) = x_1$
- If $x_2 \le x_1, u(x_1, x_2) = x_2$

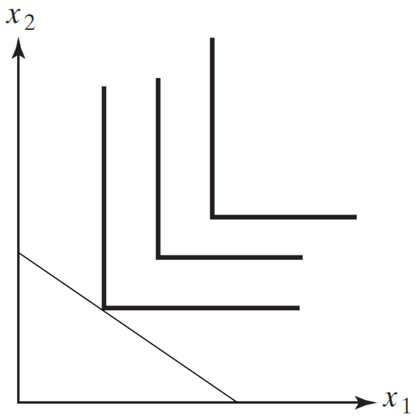
A Non-Differentiable Utility Function

- We want to find the indifference curves: the set of all (x_1, x_2) that give the same utility.
- Suppose utility is at level u^*.
  - If $x_1 \leq x_2, x_1 = u^*, x_2$ can take any value satisfying $x_1 \leq x_2$
  - If $x_2 \leq x_1, x_2 = u^*, x_1$ can take any value satisfying $x_2 \leq x_1$



- If $x_1 \leq x_2, x_1 = u^*, x_2$ can take any value satisfying $x_1 \leq x_2$
- If $x_2 \leq x_1, x_2 = u^*, x_1$ can take any value satisfying $x_2 \leq x_1$
- Is this function quasiconcave?
- Strictly quasiconcave?

Consumer Problem with Leontief Utility



- No matter what the prices p_1, p_2 are, the optimal choice will satisfy $x_1 = x_2$.

Consumer Problem with Leontief Utility

- No matter what the prices p_1, p_2 are, the optimal choice will satisfy $x_1 = x_2$.
- Plug into budget equation $p_1 x_1 + p_2 x_2 = y$, giving

$$x_1(p_1, p_2, y) = \frac{y}{p_1 + p_2}, x_2(p_1, p_2, y) = \frac{y}{p_1 + p_2}$$

- Indirect utility: plug the Marshallian demand function into the utility function:

$$v(p_1, p_2, y) = \min\left(\frac{y}{p_1 + p_2}, \frac{y}{p_1 + p_2}\right) = \frac{y}{p_1 + p_2}$$

- We can verify the properties of an indirect utility function (except Roy's Identity) apply.

Expenditure Function

- The *expenditure function* is the minimum amount of expenditure necessary to achieve a given utility level u at prices $\boldsymbol{p}$:

$$e(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

- If preferences are strictly monotonic, then the constraint will be satisfied with equality
- Denote the solution to the expenditure minimization problem as:

$$\boldsymbol{x}^h(\boldsymbol{p}, u) = \arg\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

- This is called the *Hicksian demand function* or *compensated demand*.
- It shows the effect of a change in prices on demand, while *holding utility constant*.

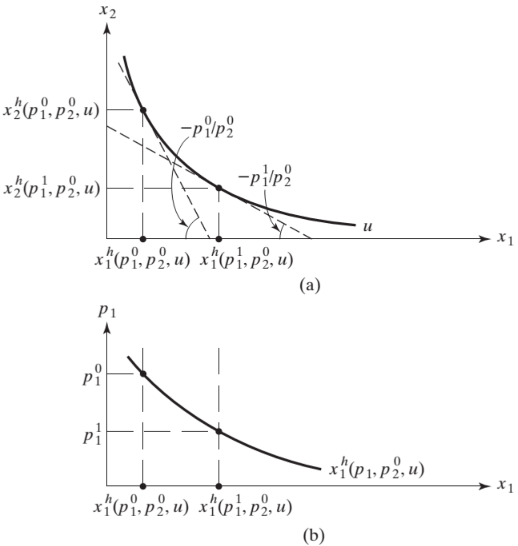


Figure 1.16. The Hicksian demand for good 1.

Properties of Expenditure Function

- If $u(\cdot)$ is continuous and strictly increasing, then $e(\boldsymbol{p}, u)$ is:
  - Zero when u is at the lowest possible level
  - Continuous
  - For all strictly positive $\boldsymbol{p}$, it is strictly increasing and unbounded above in u
  - Increasing in $\boldsymbol{p}$
  - Homogeneous of degree 1 in $\boldsymbol{p}$
  - Concave in $\boldsymbol{p}$
  - If $u(\cdot)$ is strictly quasiconcave, then it satisfies *Shephard's lemma*:

$$\frac{\partial e(\boldsymbol{p}^0, u^0)}{\partial p_i} = x_i^h(\boldsymbol{p}^0, u^0) \qquad \text{for } i = 1 \ldots n$$

Example: CES Utility

- Suppose direct utility is $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$.
- Let's derive the expenditure function:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \qquad \text{s.t.} \qquad (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- Form the Lagrangian:

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u)$$

- First-order conditions:

$$\frac{\partial L}{\partial x_1} = p1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

Example: CES Utility

$$\frac{\partial L}{\partial x_1} = p1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- Solving for x_1, x_2, we get the Hicksian demands ($r = \frac{\rho}{\rho-1}$):

$$x_1^h(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}$$

$$x_2^h(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$$

- Plug back into objective function $\boldsymbol{p} \cdot \boldsymbol{x}$:

$$e(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}$$

Indirect Utility and Expenditure Function

- Suppose we fix $(\boldsymbol{p}, y)$ and let $u = v(\boldsymbol{p}, y)$. By definition, this is the highest possible utility that can be attained given $(\boldsymbol{p}, y)$.
- Obviously, utility u can be attained given income y.
- By definition, $e(\boldsymbol{p}, u)$ is the smallest possible expenditure needed to attain u. Therefore:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, y)) \leq y$$

- Likewise, if we fix $(\boldsymbol{p}, u)$, let $y = e(\boldsymbol{p}, u)$, then expenditure y is attainable given target utility level u.

$$v(\boldsymbol{p}, e(\boldsymbol{p}, u)) \geq u$$

- These will be equalities if $u(\cdot)$ is continuous and strictly increasing.

Indirect Utility and Expenditure Function

- Theorem 1.8: Let $v(\boldsymbol{p}, y)$ and $e(\boldsymbol{p}, u)$ be the indirect utility function and expenditure function for a utility function that is continuous and strictly increasing. Then for all strictly positive $\boldsymbol{p}, y \geq 0$, and utility level u:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, y)) = y$$

$$v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$$

- This allows us to derive one from the other.

Indirect Utility and Expenditure Function

- Suppose $v(\boldsymbol{p}, y)$ is an indirect utility function for continuous, strictly increasing $u(\cdot)$.
- $v(\boldsymbol{p}, y)$ is strictly increasing in y, therefore it can be inverted to get a function that takes utility level u and gives an expenditure y:

$$v^{-1}(\boldsymbol{p} : t) = y \qquad \text{s.t. } v(\boldsymbol{p}, y) = t$$

- Apply this to both sides of $v(\boldsymbol{p}, e(\boldsymbol{p}, y)) = u$:

$$e(\boldsymbol{p}, u) = v^{-1}(\boldsymbol{p} : u)$$

- Similarly, $e(\boldsymbol{p}, u)$ is strictly increasing in u. Invert it to obtain:

$$e^{-1}(\boldsymbol{p} : t) = u \qquad \text{s.t. } e(\boldsymbol{p}, u) = t$$

- Applying to both sides of $e(\boldsymbol{p}, v(\boldsymbol{p}, y)) = y$:

$$v(\boldsymbol{p}, y) = e^{-1}(\boldsymbol{p} : y)$$

Example: CES Utility

- From before, we know that the indirect function for CES utility is:

$$v(\boldsymbol{p}, y) = y(p_1^r + p_2^r)^{-\frac{1}{r}}$$

- Suppose income is equal to $e(\boldsymbol{p}, u)$. Then

$$v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = e(\boldsymbol{p}, u)(p_1^r + p_2^r)^{-\frac{1}{r}}$$

- Using $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$, we get:

$$e(\boldsymbol{p}, u)(p_1^r + p_2^r)^{-\frac{1}{r}} = u \Rightarrow$$

$$e(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}$$

- which is the same as what we solved for directly last time.

Example: CES Utility

- Suppose we start with expenditure function instead.

$$e(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}} \Rightarrow$$

$$e(\mathbf{p}, v(\mathbf{p}, y)) = v(\mathbf{p}, y)(p_1^r + p_2^r)^{\frac{1}{r}}$$

- Using $e(\mathbf{p}, v(\mathbf{p}, y)) = y$:

$$v(\mathbf{p}, y) = y(p_1^r + p_2^r)^{-\frac{1}{r}}$$

- which is the same as before.

Relationship between Marshallian and Hicksian demand

- There is also a relationship between the solutions to these problems.
- Marshallian demand is the solution to the utility-maximization problem.
- Hicksian demand is the solution to the expenditure-minimization problem.
- Theorem 1.9: Assuming $u(\cdot)$ is continuous, strictly increasing, and strictly quasiconcave, then for strictly positive $\boldsymbol{p}, y \geq 0$, and all utility levels u:

$$x_i(\boldsymbol{p}, y) = x_i^h(\boldsymbol{p}, v(\boldsymbol{p}, y)), x_i^y(\boldsymbol{p}, y) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

- Marshallian demand at $(\boldsymbol{p}, y)$ is equal to Hicksian demand at $\boldsymbol{p}$ and the maximum possible utility achievable at $(\boldsymbol{p}, y)$.
- Hicksian demand at $\boldsymbol{p}$, utility level u is equal to Marshallian demand at $\boldsymbol{p}$ and income equal to minimum expenditure necessary to achieve u.

Relationship between Marshallian and Hicksian demand

- Proof:
  - Strict quasiconcavity of $u(\cdot)$ ensures the solution to each problem is *unique*.
  - Let $\boldsymbol{x}^0 = \boldsymbol{x}(\boldsymbol{p}^0, y^0)$ be the solution to the utility maximization problem, giving utility $u^0 = u(\boldsymbol{x}^0)$
  - Then $p^0 \cdot x^0 = y^0$ (budget constraint is satisfied with equality, due to strict monotonicity)

$$e(\boldsymbol{p}^0, v(\boldsymbol{p}^0, y^0)) = e(\boldsymbol{p}, u^0) = y$$

  - Therefore, $\boldsymbol{x}^0$ is also a solution to the expenditure minimization problem:

$$\boldsymbol{x}^0 = \boldsymbol{x}^h(\boldsymbol{p}^0, u^0)$$

$$\boldsymbol{x}(\boldsymbol{p}^0, y^0) = \boldsymbol{x}^h(\boldsymbol{p}^0, v(\boldsymbol{p}^0, y^0))$$

Example: CES Utility

- For CES utility, the Hicksian demand function is:

$$x_i^h(\boldsymbol{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_i^{r-1}, \qquad \text{for } i = 1, 2$$

- Indirect utility function is:

$$v(\boldsymbol{p}, y) = y(p_1^r + p_2^r)^{-\frac{1}{r}}$$

$$h_i^h(\boldsymbol{p}, v(\boldsymbol{p}, y)) = v(\boldsymbol{p}, y)(p_1^r + p_2^r)^{\frac{1}{r}-1} p_i^{r-1}$$

$$= y(p_1^r + p_2^r)^{-\frac{1}{r}}(p_1^r + p_2^r)^{\frac{1}{r}-1} p_i^{r-1}$$

$$= y\frac{p_i^{r-1}}{p_1^r + p_2^r}$$

- which is the same as the Marshallian demand we solved for before.

Properties of Consumer Demand (1.5)

- If preferences are as we have assumed and consumers do in fact choose by maximizing utility, this predicts that demand should satisfy certain properties.
- We can use these properties to empirically test whether observed behavior is consistent with some utility function or with optimizing behavior.
- Or, if we believe that optimizing behavior is taking place, we can use these relationships to restrict the values of parameters of the utility maximization problem.

Relative Prices and Real Income

- The *relative price* of good i to good j is simply p_i/p_j.
- *Real income* is the maximum amount of a good that can be bought with income y, so it is y/p_j.
- Utility maximization predicts that only *relative prices* and *real income* affects behavior (i.e. the amount of goods demanded).
- We can see this from the property that Marshallian demand is homogeneous of degree zero in $(\boldsymbol{p}, y)$.
- If we multiply both $\boldsymbol{p}$ and y by the same amount, demand is unchanged.

Homogeneity and Budget Balancedness

- Theorem 1.10: If $u(\cdot)$ is strictly increasing and strictly quasiconcave, then the Marshallian demand function $x_i(\boldsymbol{p}, y)$ is homogeneous of degree zero in $\boldsymbol{p}, y$, and it satisfies *budget balancedness*: $\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p}, y)$ for all $(\boldsymbol{p}, y)$.
- Homogeneity of demand is implied by homogeneity of the value function.
- Budget balancedness comes from the strictly increasing assumption; the budget constraint is always satisfied with equality.
- We can choose a good n and call it the *numeraire*, to serve as "money". All prices will be relative to the price of the numeraire good, p_n.

$$\boldsymbol{x}(\boldsymbol{p}, y) = \boldsymbol{x}\left(\frac{\boldsymbol{p}}{p_n}, \frac{y}{p_n}\right)$$

- Demand only depends on $n-1$ relative prices and real income.

Income and Substitution Effects

- We would like to know the effect on demand of a change in prices.
- Does a decrease in the price of good i result in an increase in demand for good i? Not necessarily.
- We decompose the total effect of a change in price, into the *substitution effect* and *income effect*.
- The *substitution effect* is the change in demand due to substituting the relatively cheaper good for the relatively more expensive ones.
- The *income effect* is the change due to the increase in total buying power of the consumer.

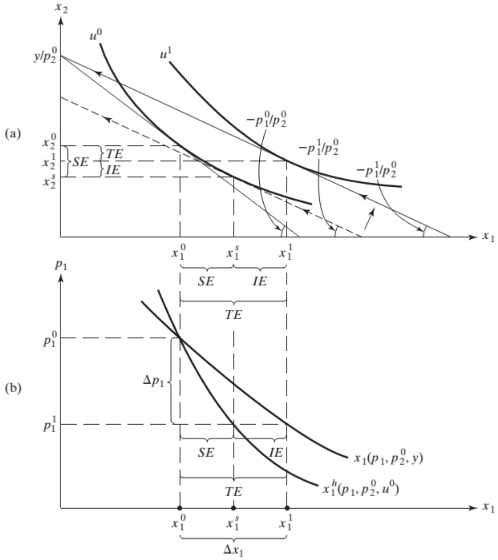


Figure 1.20. The Hicksian decomposition of a price change.

Income and Substitution Effects

- Suppose the original price is p_1^0, p_2^0, resulting in demand x_1^0, x_2^0 with utility u^0.
- Price of good 1 falls to p_1^1. Consumption of good 1 increases to x_1^1, good 2 falls to x_2^1.
- First, hypothetically allow price to fall to p_1^1 while keeping utility constant at u^0.
- This is the substitution effect.
- Then, increase income while keeping relative prices the same. This is the income effect.
- We can express this mathematically using the Slutsky equation.

Slutsky Equation

- Theorem 1.11: Let $\boldsymbol{x}(\boldsymbol{p}, y)$ be Marshallian demand, achieving utility level u^* at $(\boldsymbol{p}, y)$. Then:

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j} - x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y} \qquad \text{for } i = 1 \ldots n$$

- $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_j}$ is the total effect of a price change in good j on demand for good i.
- $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is the substitution effect.
- $x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p},y)}{\partial y}$ is the income effect.

Proof of Slutsky Equation

$$x_i^h(\boldsymbol{p}, u^*) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u^*))$$

- Differentiate both sides with respect to p_j.
- Left-hand side:
$$\frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j}$$
- Right-hand side (use chain rule):
$$\frac{x_i(\boldsymbol{p}, e(\boldsymbol{p}, u^*))}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, e(\boldsymbol{p}, u^*))}{\partial y} \frac{\partial e(\boldsymbol{p}, u^*)}{\partial p_j}$$
- Substitute $u^* = v(\boldsymbol{p}, y)$ and $e(\boldsymbol{p}, u^*) = e(\boldsymbol{p}, v(\boldsymbol{p}, y)) = y$ into the first term.
- For the second term, use
$$\frac{\partial e(\boldsymbol{p}, u^*)}{\partial p_j} = x_j^h(\boldsymbol{p}, u^*) = x_j^h(\boldsymbol{p}, v(\boldsymbol{p}, y)) = x_j(\boldsymbol{p}, y)$$

Proof of Slutsky Equation

$$\frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, y)}{\partial y} x_j(\boldsymbol{p}, y)$$

- Rearrange to get Slutsky equation.
- This decomposes any total price effect into substitution and income effects.
- However, the substitution effect may be unobservable, since we don't actually see utility levels.
- We can still deduce some properties of Hicksian demand.

Negative Own-Substitution Terms

- Theorem 1.12: Let $x_i^h(\boldsymbol{p}, u)$ be Hicksian demand for good i. Then

$$\frac{\partial x_i^h(\boldsymbol{p}, u)}{\partial p_i} \leq 0$$

- That is, Hicksian demand curves always slope downwards. If the price of good i increases, then Hicksian demand always decreases.
- This follows from the concavity of the expenditure function:

$$\frac{\partial^2 e(\boldsymbol{p}, u)}{\partial p_i^2} = \frac{\partial x_i^h(\boldsymbol{p}, u)}{\partial p_i}$$

- Second derivatives of a concave function must be non-positive.

Law of Demand

- A *normal good* is a good for which consumption increases as income increases.
- An *inferior good* is a good for which consumption decreases as income increases.
- A decrease in the *price* of a normal good will cause demand to increase.
- If an own-price decrease causes a decrease in demand, a good must be inferior. (The converse is not guaranteed).

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j} - x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y}$$

Elasticity Relations

- The *income elasticity* of demand for good i is the percentage change in x_i per 1% change in income:

$$\eta_i = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial y} \frac{y}{x_i(\boldsymbol{p}, y)}$$

- The *price elasticity* of demand for good i with respect to the price of good j is the percentage change in x_i per 1% change in the price of good j:

$$\epsilon_{ij} = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} \frac{p_j}{x_i(\boldsymbol{p}, y)}$$

- The *income share* of good i is the fraction of total income that is spent on good i:

$$s_i = \frac{p_i x_i(\boldsymbol{p}, y)}{y}, s_i \geq 0, \sum_{i=1}^{n} s_i = 1$$

Aggregation in Consumer Demand

- Theorem 1.17: Let $\mathbf{x}(\mathbf{p}, y)$ be Marshallian demand. The following relations must hold:
- Engel aggregation:

$$\sum_{i=1}^{n} s_i \eta_i = 1$$

- Cournot aggregation:

$$\sum_{i=1}^{n} s_i \epsilon_{ij} = -s_j \qquad \text{for } j = 1 \ldots n$$

- These impose conditions that must be satisfied before and after any price change.

- Exercise 1.44: In a two-good case, if one good is inferior, the other good must be normal.
- Note that an inferior good has a positive income elasticity, while a normal good has a negative elasticity.

$$\eta_i = \frac{\partial x_i(\mathbf{p}, y)}{\partial y} \frac{y}{x_i(\mathbf{p}, y)}$$

- Using the Engel aggregation condition: $\sum_{i=1}^{n} s_i \eta_i = 1$
- s_i, the share of income spent on good i, is always positive.
- If $\eta_1 < 0$, then $\eta_2 > 0$, in order to satisfy the condition $s_1\eta_1 + s_2\eta_2 = 1$.

Marshallian Demands		
Homogeneity	$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty)$	for all $(\mathbf{p}, y)$, and $t > 0$
Symmetry	$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(p, y)}{\partial y}$	
	$= \frac{\partial x_j(\mathbf{p}, y)}{\partial p_i} + x_i(\mathbf{p}, y)\frac{\partial x_j(\mathbf{p}, y)}{\partial y}$	for all $(\mathbf{p}, y)$, and $i, j = 1, \ldots, n$
Negative semidefiniteness	$\mathbf{z}^T\mathbf{s}(\mathbf{p}, y)\mathbf{z} \leq 0$	for all $(\mathbf{p}, y)$, and $\mathbf{z}$
Budget balancedness	$\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$	for all $(\mathbf{p}, y)$,
Engel aggregation	$\sum_{i=1}^n s_i\eta_i = 1$	
Cournot aggregation	$\sum_{i=1}^n s_i\varepsilon_{ij} = -s_j$	for $j = 1, \ldots, n$
Hicksian Demands		
Homogeneity	$\mathbf{x}^h(t\mathbf{p}, u) = \mathbf{x}^h(\mathbf{p}, u)$	for all $(\mathbf{p}, u)$, and $t > 0$
Symmetry	$\frac{\partial x_i^h(\mathbf{p}, y)}{\partial p_j} = \frac{\partial x_j^h(\mathbf{p}, y)}{\partial p_i}$	for $i, j = 1, \ldots, n$
Negative semidefiniteness	$\mathbf{z}^T\sigma(\mathbf{p}, u)\mathbf{z} \leq 0$	for all $\mathbf{p}$, u, and $\mathbf{z}$
Relating the Two		
Slutsky equation	$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}$	for all $(\mathbf{p}, y)$, $u = v(\mathbf{p}, y)$,
	$= \frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} - x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}$	and $i, j = 1, \ldots, n$

Figure 1.21. Properties of consumer demand.

- Consider any function of prices and utility $E(\boldsymbol{p}, u)$ that may or may not be an expenditure function.
- Suppose E satisfies the properties of an expenditure function:
- Continuity, strictly increasing, unbounded above in u
- Increasing, homogeneous of degree 1, concave, and differentiable in $\boldsymbol{p}$.
- We can show that it is, in fact, an expenditure function for some utility function.

Constructing the Utility Function

- Choose some $(\boldsymbol{p}^0, u^0)$, evaluate $E(\boldsymbol{p}^0, u^0)$ at that point.
- Construct the closed half-space in the consumption set:

$$A(\boldsymbol{p}^0, u^0) = \{\boldsymbol{x} | \boldsymbol{p}^0 \cdot \boldsymbol{x} \geq E(\boldsymbol{p}^0, u^0)\}$$

- $A(\boldsymbol{p}^0, u^0)$ is a closed, convex set containing all points on or above the hyperplane defind by $\boldsymbol{p}^0 \cdot \boldsymbol{x} = E(\boldsymbol{p}^0, u^0)$.
- Repeat the process for all prices strictly positive prices $\boldsymbol{p}$, and take the intersection of all the half-spaces:

$$A(u^0) = \bigcap_{\boldsymbol{p} >> 0} A(\boldsymbol{p}, u^0) = \{\boldsymbol{x} | \boldsymbol{p} \cdot \boldsymbol{x} \geq E(\boldsymbol{p}, u^0) \text{ for all } \boldsymbol{p} >> 0\}$$

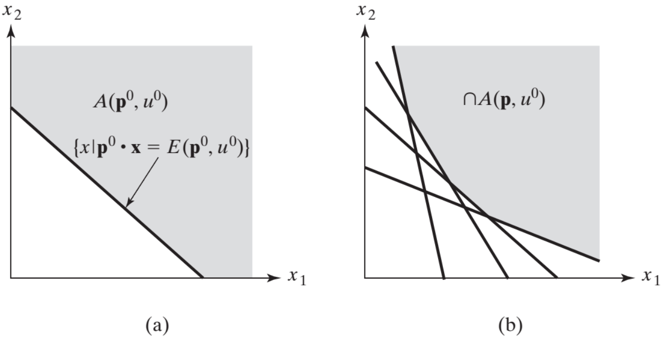


Figure 2.1. (a) The closed half-space $A(\mathbf{p}^0, u^0)$. (b) The intersection of a finite collection of the sets $A(\mathbf{p}, u^0)$.

- As the number of half-spaces increases, their intersection becomes a convex set with a smooth boundary.
- This set $A(u^0) = \bigcap_{\boldsymbol{p}>>0} A(\boldsymbol{p}, u^0)$ is an upper level set for some quasiconcave function.
- It turns out that this is a valid utility function.

Constructing the Utility Function

- Theorem 2.1: Let $E : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the properties of an expenditure function. Then the function u generated by

$$u(\boldsymbol{x}) = \max\{u \geq 0 | \boldsymbol{x} \in A(u)\}$$

- is increasing, unbounded above, and quasiconcave.
- Theorem 2.2: The Expenditure Function of u is E:
- Let $E(\boldsymbol{p}, u)$ satisfy the properties of an expenditure function, and let $u(\boldsymbol{x}$ be derived as above. Then for all non-negative prices and utility,

$$E(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

Utility Maximization and Expenditure Minimization

- There are two equivalent ways of characterizing consumer demand.
- One is to start with the direct utility function and derive Marshallian demand.
- Or, we can start with an expenditure function and use inversion and differentiation to derive demand.
- One way may be analytically simpler than the other, or may be empirically easier to observe.
- For example, we cannot directly observe utilities, but we can observe prices and expenditures.

Homework #1

- Homework #1 is due next week.
- For next week, please read Chapter 2.1 (Duality: A Closer Look) and continue to Chapter 3. We will not cover the other parts of Chapter 2.