Advanced Microeconomic Analysis, Lecture 4

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March 27, 2017

Homework #1

- Homework #1 is due at the end of class.
- I will post the solutions and HW #2 on the website.
- HW #2 is due in two class meetings.
- There is no class on April 3, it has been moved to April 1 instead.

Review of Last Week

Marshallian Demands		
Homogeneity	$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty)$	for all $(\mathbf{p}, y)$, and $t > 0$
Symmetry	$\dfrac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\dfrac{\partial x_i(p, y)}{\partial y}$	
	$= \dfrac{\partial x_j(\mathbf{p}, y)}{\partial p_i} + x_i(\mathbf{p}, y)\dfrac{\partial x_j(\mathbf{p}, y)}{\partial y}$	for all $(\mathbf{p}, y)$, and $i, j = 1, \ldots, n$
Negative semidefiniteness	$\mathbf{z}^T\mathbf{s}(\mathbf{p}, y)\mathbf{z} \leq 0$	for all $(\mathbf{p}, y)$, and $\mathbf{z}$
Budget balancedness	$\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$	for all $(\mathbf{p}, y)$,
Engel aggregation	$\sum_{i=1}^n s_i\eta_i = 1$	
Cournot aggregation	$\sum_{i=1}^n s_i\varepsilon_{ij} = -s_j$	for $j = 1, \ldots, n$
Hicksian Demands		
Homogeneity	$\mathbf{x}^h(t\mathbf{p}, u) = \mathbf{x}^h(\mathbf{p}, u)$	for all $(\mathbf{p}, u)$, and $t > 0$
Symmetry	$\dfrac{\partial x_i^h(\mathbf{p}, y)}{\partial p_j} = \dfrac{\partial x_j^h(\mathbf{p}, y)}{\partial p_i}$	for $i, j = 1, \ldots, n$
Negative semidefiniteness	$\mathbf{z}^T\sigma(\mathbf{p}, u)\mathbf{z} \leq 0$	for all $\mathbf{p}$, u, and $\mathbf{z}$
Relating the Two		
Slutsky equation	$\dfrac{\partial x_i(\mathbf{p}, y)}{\partial p_j}$	for all $(\mathbf{p}, y)$, $u = v(\mathbf{p}, y)$,
	$= \dfrac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} - x_j(\mathbf{p}, y)\dfrac{\partial x_i(\mathbf{p}, y)}{\partial y}$	and $i, j = 1, \ldots, n$

Figure 1.21. Properties of consumer demand.

- Consider any function of prices and utility $E(\boldsymbol{p}, u)$ that may or may not be an expenditure function.
- Suppose E satisfies the properties of an expenditure function:
- Continuity, strictly increasing, unbounded above in u
- Increasing, homogeneous of degree 1, concave, and differentiable in $\boldsymbol{p}$.
- We can show that it is, in fact, an expenditure function for some utility function.

Constructing the Utility Function

- Choose some $(\boldsymbol{p}^0, u^0)$, evaluate $E(\boldsymbol{p}^0, u^0)$ at that point.
- Construct the closed half-space in the consumption set:

$$A(\boldsymbol{p}^0, u^0) = \{\boldsymbol{x} | \boldsymbol{p}^0 \cdot \boldsymbol{x} \geq E(\boldsymbol{p}^0, u^0)\}$$

- $A(\boldsymbol{p}^0, u^0)$ is a closed, convex set containing all points on or above the hyperplane defind by $\boldsymbol{p}^0 \cdot \boldsymbol{x} = E(\boldsymbol{p}^0, u^0)$.
- Repeat the process for all prices strictly positive prices $\boldsymbol{p}$, and take the intersection of all the half-spaces:

$$A(u^0) = \bigcap_{\boldsymbol{p} >> 0} A(\boldsymbol{p}, u^0) = \{\boldsymbol{x} | \boldsymbol{p} \cdot \boldsymbol{x} \geq E(\boldsymbol{p}, u^0) \text{ for all } \boldsymbol{p} >> 0\}$$

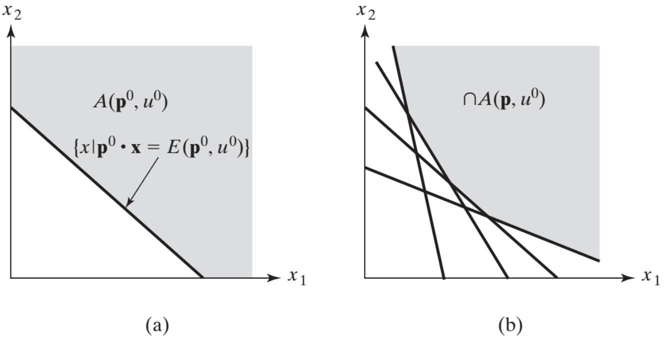

Figure 2.1. (a) The closed half-space $A(\mathbf{p}^0, u^0)$. (b) The intersection of a finite collection of the sets $A(\mathbf{p}, u^0)$.

- As the number of half-spaces increases, their intersection becomes a convex set with a smooth boundary.
- This set $A(u^0) = \bigcap_{\boldsymbol{p}>>0} A(\boldsymbol{p}, u^0)$ is an upper level set for some quasiconcave function.
- It turns out that this is a valid utility function.

Constructing the Utility Function

- Theorem 2.1: Let $E : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the properties of an expenditure function. Then the function u generated by

$$u(\boldsymbol{x}) = \max\{u \geq 0 | \boldsymbol{x} \in A(u)\}$$

- is increasing, unbounded above, and quasiconcave.
- Theorem 2.2: The Expenditure Function of u is E:
- Let $E(\boldsymbol{p}, u)$ satisfy the properties of an expenditure function, and let $u(\boldsymbol{x})$ be derived as above. Then for all non-negative prices and utility,

$$E(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

Duality Between Utility and Indirect Utility

- Duality allows us to go from the expenditure function to the direct utility function.
- Since expenditure and indirect utility functions are inverses of each other, it should be possible to go from indirect to direct utility.
- **Theorem 2.3**: Suppose $u(\mathbf{x})$ is quasiconcave and differentiable on $\mathbb{R}^n_{++}$, with strictly positive partial derivatives. Suppose the indirect utility function generated by u is $v(\mathbf{p}, y)$. Then for all $\mathbf{x} \in \mathbb{R}^n_{++}$:

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_{++}} v(\mathbf{p}, \mathbf{p} \cdot \mathbf{x})$$

- An equivalent way, which may be simpler, of obtaining u is with the problem

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_{++}} v(\mathbf{p}, 1) \qquad \text{subject to } \mathbf{p} \cdot \mathbf{x} = 1$$

Example 2.1: CES Utility from Indirect Utility

- Suppose we are given the indirect utility function

$$v(p_1, p_2, y) = y(p_1^r + p_2^r)^{\frac{-1}{r}}$$

- Let's find the direct utility function that generates this indirect utility. Set $y = 1$, then $v(p_1, p_2, 1) = (p_1^r + p_2^r)^{-\frac{1}{r}}$.

$$u(x_1, x_2) = \min p_1, p_2 (p_1^r + p_2^r)^{\frac{-1}{r}} \qquad \text{s.t. } p_1 x_1 + p_2 x_2 - 1 = 0$$

$$L(p_1, p_2, \lambda) = (p_1^r + p_2^r)^{\frac{-1}{r}} - \lambda(p_1 x_1 + p_2 x_2 - 1)$$

Example 2.1: CES Utility from Indirect Utility

- First order conditions:

$$\frac{\partial L}{\partial p_1} = -(p_1^r + p_2^r)^{\frac{-1}{r}-1} p_1^{r-1} - \lambda x_1 = 0$$

$$\frac{\partial L}{\partial p_2} = -(p_1^r + p_2^r)^{\frac{-1}{r}-1} p_2^{r-1} - \lambda x_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - 1 = 0$$

- Solving this system of equations for $u(x_1, x_2)$ gives us

$$u(x_1, x_2) = (x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}})^{\frac{r-1}{r}}$$

- which is the original CES utility function, with $\rho = r/(r-1)$.

Utility Maximization and Expenditure Minimization

- There are two equivalent ways of characterizing consumer demand.
- One is to start with the direct utility function and derive Marshallian demand.
- Or, we can start with an expenditure function and use inversion and differentiation to derive demand.
- One way may be analytically simpler than the other, or may be empirically easier to observe.
- For example, we cannot directly observe utilities, but we can observe prices and expenditures.

Chapter 2.4: Risk and Uncertainty

- So far, we've studied consumer choice in a deterministic situation.
- Now, we introduce *uncertainty*.
- Instead of consumers choosing among bundles of goods, the consumer (here, called an *agent*) will choose among *gambles* (also called *lotteries*).
- A *gamble* is a probability distribution over a finite set of outcomes.

Gambles

- Let $A = \{a_1, \ldots, a_n\}$ denote a finite set of *outcomes*.
- A *simple gamble* on A assigns a probability p_i to each outcome $a_i \in A$, denoted:

$$(p_1 \circ a_1, \ldots, p_n \circ a_n)$$

- where each $p_i \geq 0$, and the sum of all probabilities is 1.
- For example, suppose I offer to flip a coin, and pay you 1 if it is heads, and pay you -1 if it is tails.
- The set of outcomes in this case is $\{1, -1\}$.
- Assuming the coin is fair, the probability of each outcome is $\frac{1}{2}$.
- The *simple gamble* corresponding to this situation is

$$\left(\frac{1}{2} \circ 1, \frac{1}{2} \circ -1\right)$$

Set of Simple Gambles

- The set of all possible simple gambles on $A = \{a_1, ..., a_n\}$ is denoted as:

$$G_S = \{(p_1 \circ a_1, ..., p_n \circ a_n) | p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$$

- A *degenerate gamble* is a gamble that assigns probability 1 to some outcome a_i, i.e. offers a_i with certainty.
- A *compound gamble* is a gamble with an outcome that is another gamble.
- For example: I offer to flip a coin. If it is heads, I pay you 1, and if it is tails, I offer the gamble in the previous slide.
- We will only deal with gambles with finitely many layers (though it is possible to define infinitely layered gambles).
- Let G denote the set of *all* possible finite gambles on A, both simple and compound.
- The agent will have *preferences* over gambles in G.

Axiom 1 & 2: Completeness, Transitivity

- As in consumer theory, we will state *axioms* that any reasonable preference relation must satisfy.
- Axiom 1: Completeness
  - For any two gambles g, g' in G, either $g \succsim g'$ is true, $g' \succsim g$ is true, or both.
- Axiom 2: Transitivity
  - For any three gambles g, g', g'' in G, if $g \succsim g'$ and $g' \succsim g''$, then $g \succsim g''$.
- Consider the n degenerate gambles that offer outcome a_i with certainty, $(1 \circ a_i)$.
- By Axiom 1, given any pair g, g' of these gambles, at least one is preferable to the other.
- A homework problem asks you to prove that all of these gambles must be ordered by $\succsim$.

Axiom 3: Continuity

- Assume that we order the outcomes by preference, so that $a_1 \succsim a_2 \succsim \ldots \succsim a_n$.
- Axiom 3: Continuity
  - For any gamble g in G, there is some probability $\alpha \in [0,1]$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$.
- That is, given any gamble g, there is some combination of the *best* and *worst* possible outcomes that is indifferent to g.
- For example, suppose $A = \{1000, 10, \text{death}\}$, with preferences over these outcomes:

$$1000 \succ 10 \succ \text{death}$$

- This axiom states that there is some probability α that makes the gamble $(\alpha \circ 1000, (1-\alpha) \circ \text{death})$ indifferent to 10.

Axiom 4: Monotonicity

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  - For all probabilities $\alpha, \beta \in [0,1], \alpha \geq \beta$ if and only if

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$$

- That is, if two simple gambles offer only the *best* and *worst* outcomes, the gamble with the higher probability on the best outcome is preferred.
- This implies that $(1 \circ a_1, 0 \circ a_n) \succ (0 \circ a_1, 1 \circ a_n)$, that is, among degenerate gambles, $a_1 \succ a_n$.
- Indifference among all outcomes in A is ruled out.

Axiom 5: Substitution

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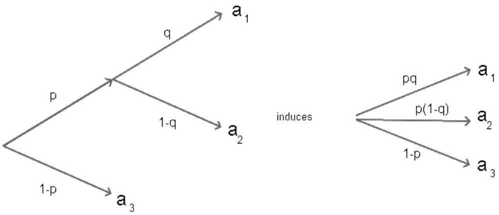
  - If $g = (p_1 \circ g^1, ...p_k \circ g^k)$ and $h = (p_1 \circ h^1, ...p_k \circ h^k)$ are gambles in G, and $h^i \sim g^i$ for every i, then $h \sim g$.
- The agent is indifferent between gambles if their *realizations* (i.e. an outcome that may be another gamble) are indifferent, and the probability on each realization is the same.
- Suppose $g \sim h$. By Axiom 1, we know that $g \sim g$. Therefore, every convex combination of g, h is indifferent to g and h:

$$(\alpha \circ g, (1-\alpha) \circ h) \sim (\alpha \circ g, (1-\alpha) \circ g) \sim g \qquad \text{for } 0 \leq \alpha \leq 1$$

Inducing a Simple Gamble



- Suppose g is a compound gamble. We can calculate the effective probability assigned to each outcome a_i by calculating all paths that lead to a_i, and summing the probability of each path.
- We say that g *induces the simple gamble* $(p_1 \circ a_1, ..., p_n \circ a_n)$.

Axiom 6: Reduction to Simple Gambles

- Axiom 6: Reduction to Simple Gambles
  - For any gamble $g \in G$, if $(p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple gamble induced by g, then $(p_1 \circ a_1, \ldots, p_n \circ a_n) \sim g$
- By transitivity, an agent's preferences over *all* gambles are completely determined by preferences over *simple* gambles.

von Neumann-Morgenstern Utility Functions

- As before, we would like to find a utility function $u(\cdot)$ that represents the preferences $\succsim$, i.e.

$$u(g) \geq u(g') \qquad \text{iff} \qquad g \succsim g'$$

- When we defined preferences over bundles, only three axioms (completeness, transitivity, continuity) were sufficient to guarantee existence of a continuous utility function.
- Now, we have assumed additional axioms, so the properties of utility functions should be more restricted.
- It turns out that a utility function that represents preferences satisfying these axioms has the *expected utility* property.
- Utility functions satisfying this property are called *von Neumann-Morgenstern*, or VNM, utility functions.

Expected Utility Property

- Suppose $u : G \to \mathbb{R}$ is a utility function representing $\succsim$. We will say $u(a_i)$ when we really mean $u(1 \circ a_i)$.
- $u(\cdot)$ has the *expected utility* property if, for every gamble $g \in G$:

$$u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

- where $(p_1 \circ a_1, ..., p_n \circ a_n)$ is the simple gamble induced by g.
- That is, utility of a gamble g is equal to the expected value of utilities on the outcomes a_i, where the the probability on a_i is induced by g.
- For a simple gamble $(p_1 \circ a_1, ..., p_n \circ a_n)$, its utility must be

$$u(p_1 \circ a_1, ..., p_n \circ a_n) = \sum_{i=1}^{n} p_i u(a_i)$$

- Therefore, $u(\cdot)$ is completely determined by the values it assumes on the finite set of outcomes A.

Existence of a VNM Utility Function

- We will construct a VNM utility function that represents a preference $\succsim$ satisfying our axioms.
- Consider an arbitrary gamble $g \in G$. Define $u(g)$ to be the number satisfying
$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n)$$
- By the continuity axiom, this number (a probability) must exist.
- Now, we have a candidate for a utility function. We will show it represents $\succsim$, and that it satisfies the expected utility property.

Proof that $u(\cdot)$ represents $\succsim$

- We want to show: $g \succsim g'$ iff $u(g) \geq u(g')$.
- By the monotonicity axiom: $u(g) \geq u(g')$ iff

$$(u(g) \circ a_1, (1 - u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1 - u(g')) \circ a_n)$$

- By the definition of $u(\cdot)$:
  - $(u(g) \circ a_1, (1 - u(g)) \sim g$
  - $(u(g') \circ a_1, (1 - u(g')) \circ a_n) \sim g'$
- By transitivity, $u(g) \geq u(g')$ iff $g \succsim g'$.

Proof of Expected Utility Property

- Let g be any gamble, that induces the simple gamble $g_s = (p_1 \circ a_1, \ldots, p_n \circ a_n)$.
- We want to show: $u(g) = \sum_{i=1}^{n} p_i u(a_i)$
- By the reduction to simple gambles axiom, $g \sim g_s$, therefore $u(g) = u(g_s)$.
- Let q^i denote the simple gamble $u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n)$, which is $\sim a_i$.
- Let g' denote the compound gamble $(p_1 \circ q^1, \ldots, p_n \circ q^n)$, which is $\sim g_s$ by the substitution axiom.
- We want to find the simple gamble induced by g'. This is

$$g'_s = \left((\sum_{i=1}^{n} p_i u(a_i)) \circ a_1, (1 - \sum_{i=1}^{n} p_i u(a_i)) \circ a_n \right)$$

- Definition of $u(\cdot)$ and expected utility property are satisfied with:

$$u(g_s) = \sum_{i=1}^{n} p_i u(a_i)$$

Construction of Expected Utility

- This procedure lets us construct the expected utility function, if we know the probability that makes the agent indifferent between an outcome and a best-worst gamble.
- To determine the utility of any outcome a_i, ask the agent for the probability on a_1, the best outcome, that would make him indifferent between a_i and $(\alpha \circ a_1, (1-\alpha) \circ a_n)$.
- Repeat this for every outcome $a_i \in A$, then we can calculate the expected utility for any gamble.

Example 2.4

- Suppose $A = \{10, 4, -2\}$, where the best outcome is 10 and the worst is -2.
- For each outcome a_i, we ask the agent to give a probability α that makes him indifferent between a_i and $(\alpha \circ 10, (1-\alpha) \circ -2)$.
- Suppose the answer is:
  - $10 \sim (1 \circ 10, 0 \circ -2) \rightarrow u(10) = 1$
  - $4 \sim (0.6 \circ 10, 0.4 \circ -2) \rightarrow u(4) = 0.6$
  - $-2 \sim (0 \circ 10, 1 \circ -2) \rightarrow u(-2) = 0$
- The utility of the best outcome is always 1 and the worst outcome is always 0.
- Now, we can calculate the utility of any gamble, by calculating the expected utility.
- Note that the expected outcome of the second gamble is 5.2, yet the agent is indifferent between that and the certain outcome 4.

Positive Affine Transformations

- Is a VNM utility invariant to an increasing transformation, like an ordinary utility function?
- Only if the transformation preserves the expected utility property.
- Theorem 2.8: Suppose the VNM utility function $u(\cdot)$ represents $\succsim$. Then the transformed VNM utility $v(\cdot)$ represents the same preferences if and only if:

$$v(g) = \alpha + \beta u(g)$$

- for some scalars α, β, where $\beta > 0$.

Risk Aversion

- We will examine an agent's attitude towards risk.
- Gambles will be over different levels of *wealth*.
- The set of outcomes, A, is the set of non-negative wealth levels, $\mathbb{R}_+$.
- Note that A is an infinite set. We will only consider gambles where a finitely many number of outcomes have positive probability.
- A simple gamble is of the form $(p_1 \circ w_1, ...p_n \circ w_n)$, where all w_i's are non-negative numbers.
- Assume the VNM utility function $u(\cdot)$ is differentiable and increasing, with $u'(w) > 0$ for all wealth levels w.

Risk Aversion

- The expected value of the simple gamble g offering outcomes $w_1, \ldots, w_n$ with probabilities $p_1, \ldots, p_n$ is:

$$E(g) = \sum_{i=1}^{n} p_i w_i$$

- Suppose the agent is given a choice between gamble g, and the certain outcome $E(g)$.
- The utility of each choice is:

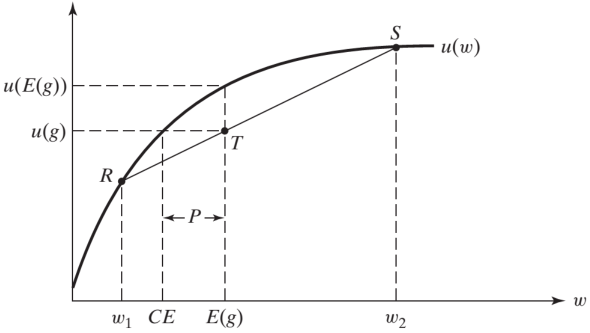
$$u(g) = \sum_{i=1}^{n} p_i u(w_i), u(E(g)) = u(\sum_{i=1}^{n} p_i w_i)$$

- The first is the VNM utility of the gamble g; the second is the VNM utility of the gamble's expected value.
- The agent prefers the choice with the higher expected utility.

Risk Aversion/Neutrality/Loving

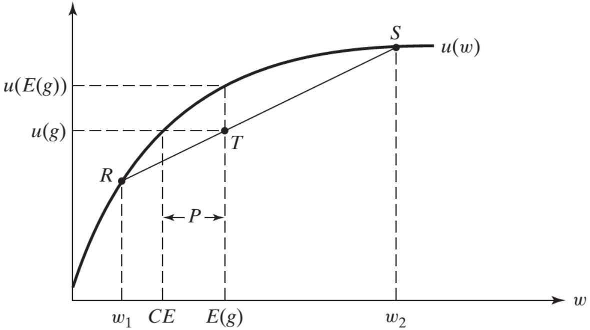
- If $u(E(g)) > u(g)$, we say the agent is *risk averse* at g.
- If $u(E(g)) = u(g)$, we say the agent is *risk neutral* at g.
- If $u(E(g)) < u(g)$, we say the agent is *risk loving* at g.
- If the agent is risk averse at *every* non-degenerate, simple gamble g, then we say the agent is *risk averse* (likewise for risk-neutral, risk-loving).
- An agent is risk averse if and only if his VNM utility function is strictly concave.
- An agent is risk neutral if and only if his VNM utility function is linear.
- An agent is risk loving if and only if his VNM utility function is strictly convex.

Risk Averse Utility Function



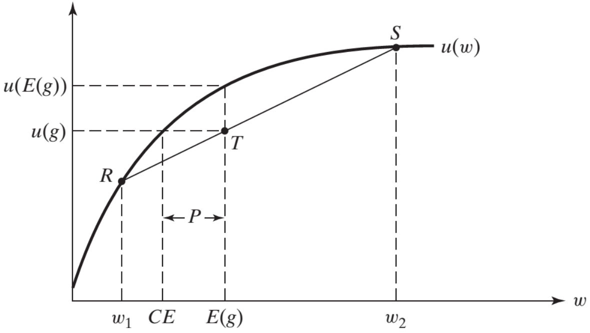
- Suppose the gamble $g = (p \circ w_1, (1-p) \circ w_2)$.
- The agent is offered a choice between receiving $E(g) = pw_1 + (1-p)w_2$ with certainty, or the gamble g.
- $u(g) = pu(w_1) + (1-p)u(w_2), u(E(g)) = u(pw_1 + (1-p)w_2$
- Here, $T = pR + (1-p)S$.

Risk Averse Utility Function



- $T = (E(g), u(E(g)))$. Since $u(E(g)) > u(g)$, the agent is risk averse.
- There is some amount of wealth we could offer, with certainty, that would be indifferent with g. This is called the *certainty equivalent* of the gamble g.
- For a risk-averse agent, the certainty equivalent is less than $E(g)$.

Risk Averse Utility Function



- There is some amount of wealth we could offer, with certainty, that would be indifferent with g. This is called the *certainty equivalent* of the gamble g.
- For a risk-averse agent, the certainty equivalent is less than $E(g)$.
- In effect, the agent is willing to pay to avoid the gamble. The willingness to pay to avoid risk is measured by the *risk premium*.

Certainty Equivalent and Risk Premium

- The *certainty equivalent* of any simple gamble g is an amount of wealth, CE, such that $u(g) = u(CE)$.
- The *risk premium* is an amount of wealth, P, such that $u(g) = u(E(g) - P)$.
- $P = E(g) - CE$.

Example 2.5

- Suppose $u(w) = \ln(w)$. $u(\cdot)$ is strictly concave, therefore risk averse.
- Let g be a gamble that offers a 50-50 chance of winning or losing h. If initial wealth is w_0, the gamble is:

$$g = \left(\frac{1}{2} \circ (w_0 + h), \frac{1}{2} \circ (w_0 - h)\right)$$

- $E(g) = w_0$. The certainty equivalent must satisfy:

$$\ln(CE) = \frac{1}{2}\ln(w_0 + h) + \frac{1}{2}\ln(w_0 - h)$$

$$= \ln((w_0 + h)^{\frac{1}{2}}(w_0 - h)^{\frac{1}{2}}) = \ln(w_0^2 - h^2)^{\frac{1}{2}}$$

- $CE = (w_0^2 - h^2)^{\frac{1}{2}} < E(g)$ and $P = w_0 - (w_0^2 - h^2)^{\frac{1}{2}} > 0$

Arrow-Pratt Measure of Risk Aversion

- We would like to quantify how risk-averse an agent is.
- Since risk aversion is related to concavity, a more risk-averse agent should have a "more concave" utility function.
- The Arrow-Pratt measure of *absolute risk aversion* is:

$$R_a(w) = \frac{-u''(w)}{u'(w)}$$

- If the sign of $R_a(w)$ is positive/negative/zero, the agent is risk-averse/loving/neutral at w.
- Any positive affine transformation of utility leaves $R_a(w)$ unchanged:
  - Adding a constant has no effect on the numerator or denominator.
  - Multiplying by a constant leaves the ratio unchanged.

Arrow-Pratt Measure of Risk Aversion

- We want to show that a higher $R_a(w)$ means the agent has a lower CE and accepts fewer gambles.
- Suppose there are two agents with VNM utility functions $u(w), v(w)$.
- Assume that agent 1's measure of risk aversion is greater at every w:

$$R_a^1(w) = \frac{-u''(w)}{u'(w)} > \frac{-v''(w)}{v'(w)} = R_a^2(w) \qquad \text{for all } w \geq 0$$

- Define $h : [0, \infty) \Rightarrow \mathbb{R}$ as follows:

$$h(x) = u(v^{-1}(x)) \qquad \text{for all } x \geq 0$$

Arrow-Pratt Measure of Risk Aversion

$$h(x) = u(v^{-1}(x)) \qquad \text{for all } x \geq 0$$

- Using the derivative of an inverse function: $\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$, we get:

$$h'(x) = \frac{u'(v^{-1}(x))}{v'(v^{-1}(x))}$$

$$h''(x) = \frac{u'(v^{-1}(x))\left[\frac{u''(v^{-1}(x))}{u'(v^{-1}(x))} - \frac{v''(v^{-1}(x))}{v'(v^{-1}(x))}\right]}{\left[v'(v^{-1}(x))\right]^2}$$

Arrow-Pratt Measure of Risk Aversion

$$h'(x) = \frac{u'(v^{-1}(x))}{v'(v^{-1}(x))} > 0$$

- since $u', v' > 0$.

$$h''(x) = \frac{u'(v^{-1}(x))\left[\frac{u''(v^{-1}(x))}{u'(v^{-1}(x))} - \frac{v''(v^{-1}(x))}{v'(v^{-1}(x))}\right]}{\left[v'(v^{-1}(x))\right]^2} < 0$$

- since $R_a^1(w) > R_a^2(w)$ by assumption.
- Therefore, h is a strictly increasing, strictly concave function.

Jensen's Inequality

- Suppose $f(\cdot)$ is concave, $x_1 \ldots x_n$ are numbers in the domain of f, and $a_1 \ldots a_n$ are positive weights. Then:

$$f\left(\frac{\sum a_i x_i}{\sum a_j}\right) \geq \frac{\sum a_i f(x_i)}{\sum a_j}$$

- with a strict inequality if $f(\cdot)$ is strictly concave.
- If a_i's are probabilities that sum to 1, then

$$f\left(\sum a_i x_i\right) \geq \sum a_i f(x_i)$$

Arrow-Pratt Measure of Risk Aversion

- Consider the gamble $(p_1 \circ w_1, ..., p_n \circ w_n)$.
- Let $\hat{w}_1, \hat{w}_2$ denote each agent's certainty equivalent for this gamble.

$$\sum_{i=1}^{n} p_i u(w_i) = u(\hat{w}_1), \sum_{i=1}^{n} p_i v(w_i) = v(\hat{w}_2)$$

- We will show $\hat{w}_1 < \hat{w}_2$.
- Using $h(x) = u(v^{-1}(x)) \Rightarrow h(v(w)) = u(w)$:

$$u(\hat{w}_1) = \sum_{i=1}^{n} p_i u(w_i) = \sum_{i=1}^{n} p_i h(v(w_i)) < h\left(\sum_{i=1}^{n} p_i v(w_i)\right)$$

- The inequality is by Jensen's Inequality on a strictly concave function.

$$h\left(\sum_{i=1}^{n} p_i v(w_i)\right) = h(v(\hat{w}_2)) = u(\hat{w}_2)$$

Arrow-Pratt Measure of Risk Aversion

$$h\left(\sum_{i=1}^{n} p_i v(w_i)\right) = h(v(\hat{w}_2)) = u(\hat{w}_2)$$

- We have $u(\hat{w}_1) < u(\hat{w}_2)$, therefore $\hat{w}_1 < \hat{w}_2$ since u is strictly increasing.
- Note that $u(w) = h(v(w))$, where h is strictly concave. u is a "concavification" of v.

Risk Aversion as a Function of Wealth

- $R_a(w)$ is a *local* measure of risk aversion, so it may not be the same at all levels of w.
- We can classify VNM utility functions by how $R_a(w)$ varies as w increases.
- A VNM utility function displays:
  - *constant* absolute risk aversion (CARA) if $R_a(w)$ is constant as w increases;
  - *decreasing* absolute risk aversion (DARA) if $R_a(w)$ is decreasing as w increases;
  - *increasing* absolute risk aversion (IARA) if $R_a(w)$ is increasing as w increases.
- DARA is commonly used, and makes intuitive sense: a billionaire should be less risk-averse than a poor person, given the same gamble.
- CARA: as wealth increases, there is no change in willingness to accept the same gamble. IARA: there is a decrease in willingness to accept the same gamble.

Example 2.6

- Consider a risk-averse investor who decides how much of initial wealth w to invest in a risky asset.
- The risky asset's return is a random variable, with possible outcomes r_i and probability p_i for $i = 1 \ldots n$. If 1 unit is invested in the asset today, $(1 + r_i)$, a random variable, will be returned next period.
- Suppose $0 \leq \beta \leq w$ is the amount of wealth to be invested in the risky asset.
- Final wealth under outcome i will be: $(w - \beta) + (1 + r_i)\beta = w + \beta r_i$.
- Investor's problem: choose β to maximize expected utility of final wealth.

$$\max_{\beta} \sum_{i=1}^{n} p_i u(w + \beta r_i) \qquad \text{s.t. } 0 \leq \beta \leq w$$

Example 2.6

$$\max_{\beta} \sum_{i=1}^{n} p_i u(w + \beta r_i) \qquad \text{s.t. } 0 \le \beta \le w$$

- First, let's determine the conditions under which *zero* wealth is invested in the risky asset, i.e. $\beta^* = 0$.
- This is a corner solution. The objective function must be non-increasing at $\beta^* = 0$, therefore the derivative with respect to β must be ≤ 0.

$$\frac{\partial u}{\partial \beta} = \sum_{i=1}^{n} p_i u'(w + \beta r_i) r_i = u'(w) \sum_{i=1}^{n} p_i r_i \le 0$$

- Since $u'(w)$ is always positive, by assumption, then the expected return, $\sum_{i=1}^{n} p_i r_i$ must be ≤ 0.
- A risk-averse agent will invest $\beta = 0$ in the risky asset if and only if the asset's expected return is non-positive.
- Equivalently, we say that a risk-averse investor will always invest $\beta > 0$ in a risky asset with a strictly positive expected return.

Example 2.6

$$\max_{\beta} \sum_{i=1}^{n} p_i u(w + \beta r_i) \qquad \text{s.t. } 0 \leq \beta \leq w$$

- Assume the risky asset has a positive expected return (therefore, we rule out $\beta^* = 0$).
- Assume $\beta^* < w$ (i.e. not all wealth is invested).
- First-order conditions:

$$\sum_{i=1}^{n} p_i u'(w + \beta r_i) r_i = 0$$

- Second-order conditions:

$$\sum_{i=1}^{n} p_i u''(w + \beta r_i) r_i^2 < 0$$

- where the inequality is due to the strict concavity of $u(\cdot)$.
- What happens to β^* as w increases?

Example 2.6

$$\sum_{i=1}^{n} p_i u'(w + \beta^*(w) r_i) r_i = 0$$

- β^* is a function of w; take derivative with respect to w.

$$\sum_{i=1}^{n} p_i r_i \left[u''(w + \beta^*(w) r_i) \left(1 + \frac{\partial \beta^*}{\partial w} r_i \right) \right] = 0$$

$$\sum_{i=1}^{n} p_i r_i u''(w + \beta^*(w) r_i) + \frac{\partial \beta^*}{\partial w} \sum_{i=1}^{n} p_i r_i^2 u''(w + \beta^*(w) r_i) = 0$$

$$\frac{\partial \beta^*}{\partial w} = \frac{-\sum_{i=1}^{n} p_i r_i u''(w + \beta^*(w) r_i)}{\sum_{i=1}^{n} p_i r_i^2 u''(w + \beta^*(w) r_i)}$$

- The denominator is negative, as we saw in the previous slide.
- If the numerator is negative, then the risky asset is a normal good: demand increases with wealth.
- We show that DARA is sufficient to ensure this.

Example 2.6

- DARA implies:

$$R_a(w) > R_a(w + \beta^* r_i) \qquad \text{if } r_i > 0$$

$$R_a(w) < R_a(w + \beta^* r_i) \qquad \text{if } r_i < 0$$

$$R_a(w) r_i > R_a(w + \beta^* r_i) r_i$$

- From definition of $R_a(w)$:

$$R_a(w) r_i > \frac{-u''(w + \beta^* r_i)}{u'(w + \beta^* r_i)} r_i$$

$$R_a(w) r_i u'(w + \beta^* r_i) > -u''(w + \beta^* r_i) r_i$$

- Taking expectations of both sides:

$$R_a(w) \sum_{i=1}^{n} p_i r_i u'(w + \beta^* r_i) = 0 > -\sum_{i=1}^{n} p_i r_i u''(w + \beta^* r_i)$$

Example 2.7

- A risk-averse agent with initial wealth w_0, VNM utility $u(\cdot)$ chooses how many units, x, of car insurance to buy.
- Suppose there is only one type of accident, that causes a loss of L, and occurs with probability α.
- Insurance is an asset that pays 1 per unit if an accident occurs, and 0 otherwise (therefore, it is a risky asset).
- Let ρ be the price of one unit of insurance. Assume that the price is *actuarially fair*, that is, the seller of insurance makes zero expected profit.
- Expected profit per unit is $\alpha(\rho - 1) + (1 - \alpha)\rho = 0$, therefore $\rho = \alpha$.

Example 2.7

- The agent's problem:

$$\max_x \alpha u(w_0 - \alpha x - L + x) + (1-\alpha) u(w_0 - \alpha x)$$

- First-order conditions:

$$(1-\alpha)\alpha u'(w_0 - \alpha x - L + x) - \alpha(1-\alpha) u'(w_0 - \alpha x) = 0$$

$$u'(w_0 - \alpha x - L + x) = u'(w_0 - \alpha x)$$

- By assumption of risk aversion, u' is strictly decreasing. Therefore, if $u'(w_1) = u'(w_2)$, then $w_1 = w_2$:

$$w_0 - \alpha x - L + x = w_0 - \alpha x \Rightarrow x = L$$

- Therefore, if the price of insurance is actuarially fair, a risk-averse agent *fully insures* against risk: wealth is the same, $w_0 - \alpha L$, whether the accident happens or not.

Homework #1

- Homework #1 is due at the end of class.
- I will post the solutions and HW #2 on the website.
- HW #2 is due in two class meetings.
- There is no class on April 3, it has been moved to April 1 instead.