Advanced Microeconomic Analysis, Lecture 5

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Homework #2

- Homework #2 is due next meeting.
- The midterm will be on April 17.
- Midterm will be open-book.
- Chapters 1, 2.1, 2.4, 3, and possibly 4 will be covered.
- Sample midterms from previous semesters are on the course website.

Review of Last Week

- Previously, we studied consumer behavior under *deterministic* conditions.
- We want to study choice when there is *uncertainty*, that is, things like wealth, income, payoffs, etc. may be random.
- A *gamble* (or a *lottery*) is a probability distribution over a finite set of outcomes.
- $(p_1 \circ a_1, ..., p_n \circ a_n)$ means that outcome a_i will occur with probability p_i, where $p_i > 0, \sum_i p_i = 1$.
- We can axiomatically define preferences over gambles, just like we did for market bundles.

Review of Last Week

- Preferences that satisfy our axioms can be represented by a utility function that satisfies the *expected utility* property.
- The utility received from a gamble $(p_1 \circ a_1, ..., p_n \circ a_n)$ is equal to the weighted average of utilities received from *certain outcomes* $a_1, ..., a_n$:

$$u(p_1 \circ a_1, ..., p_n \circ a_n) = \sum_i p_i u(a_i)$$

- Utility functions that satisfy this property are called von Neumann-Morgenstern, or VNM, utility functions.

Review of Last Week

- Now, consider VNM utility functions over *wealth*.
- Suppose a consumer is offered a gamble $g = (p_1 \circ w_1, ..., p_n \circ w_n)$, where w_i is the consumer's wealth in outcome i.
- The expected wealth of this gamble, $E(g)$, is $\sum_i p_i w_i$.
- The shape of a VNM utility function determines the consumer's *attitude towards risk*.
- A consumer with a *concave* utility function is *risk-averse*, and gets a higher utility from receiving $E(g)$ with 100% probability, than from the uncertain gamble g.
- A consumer with a *linear* utility function is *risk-neutral*, and gets the same utility from receiving $E(g)$ with 100% probability, and the uncertain gamble g.
- A consumer with a *convex* utility function is *risk-loving*, and gets a lower utility from receiving $E(g)$ with 100% probability, than from the uncertain gamble g.

- Suppose $u(w) = -\exp(-\rho w)$, where ρ is a positive constant. Let's find the Arrow-Pratt measure of risk aversion, and solve a portfolio choice problem.

$$
\begin{aligned}
R_a(w) &= -\frac{u''(w)}{u'(w)} = -\frac{-\exp(-\rho w)\rho^2}{\exp(-\rho w)\rho} \\
&= \rho
\end{aligned}
$$

- The measure of ARA is a constant, hence exponential utility is also called CARA utility.
- The parameter ρ determines the risk aversion of the agent.

- Suppose a consumer with $u(w) = -\exp(-\rho w)$ has initial wealth w_0, and must choose a fraction $x, 0 \leq x \leq 1$ of his wealth to invest in a risky asset.
- The risky asset has two outcomes. If an amount xw_0 is invested, then:
  - with probability p, it will be a total loss (return zero).
  - with probability $1-p$, it will double the investment (return $2xw_0$).
- Let's calculate the final wealth w_1 in each outcome, expected wealth, and expected utility.
- In the "bad" outcome, with probability p, final wealth will be $w_1 = (1-x)w_0$.
- In the "good" outcome, with probability $1-p$, final wealth will be $w_1 = (1-x)w_0 + 2xw_0 = (1+x)w_0$.

- Expected wealth is:

$$
\begin{aligned}
E(w_1) &= p((1-x)w_0) + (1-p)((1+x)w_0) \\
&= (1+x-2px)w_0
\end{aligned}
$$

- Expected utility is:

$$
\begin{aligned}
E(u(w_1)) &= pu((1-x)w_0) + (1-p)u((1+x)w_0) \\
&= p - \exp(-\rho(1-x)w_0) + (1-p) - \exp(-\rho(1+x)w_0)
\end{aligned}
$$

- Expected utility is:

$$E(u(w_1)) = pu((1-x)w_0) + (1-p)u((1+x)w_0)$$
$$= -p\exp(-\rho(1-x)w_0) + -(1-p)\exp(-\rho(1+x)w_0)$$

- To find the optimal x, let's maximize expected utility:

$$\max_x E(u(w_1)) = \max_x -p\exp(-\rho(1-x)w_0) - (1-p)\exp(-\rho(1+x)w_0)$$

- First order conditions are:

$$\frac{\partial E(u(w_1))}{\partial x} = -p\rho w_0 \exp(-\rho(1-x)w_0) + (1-p)\rho w_0 \exp(-\rho(1+x)w_0) = 0$$
$$(1-p)\rho w_0 \exp(-\rho(1-x)w_0) = p\rho w_0 \exp(-\rho(1+x)w_0)$$
$$\frac{p}{1-p} = \frac{\rho w_0 \exp(-\rho(1-x)w_0)}{\rho w_0 \exp(-\rho(1+x)w_0)}$$
$$= \exp(-\rho w_0(-2x))$$
$$x^* = \frac{\log(\frac{1-p}{p})}{2\rho w_0}$$

$$x^* = \frac{\log(\frac{1-p}{p})}{2\rho w_0}$$

- The optimal fraction x^* invested in the risky asset is increasing in $(1 - p)$, the probability of the "good" outcome, and decreasing in p, the probability of the "bad" outcome.
- x^* is decreasing in ρ, the measure of absolute risk aversion.
- The absolute amount invested in the risky asset is $x^* w_0 = \frac{\log(\frac{1-p}{p})}{2\rho}$, which is a constant that does not depend on w_0.
- This is a consequence of CARA utility: there is no "wealth effect", that is, consumer's demand is the same in absolute terms, regardless of initial wealth.
- Other utility functions will have wealth effects: demand will depend on initial wealth.

Chapter 3: Theory of the Firm

- What is a firm? Here, a firm is an entity that:
  - acquires inputs;
  - combines them to produce outputs;
  - sells outputs on the market.
- The firm's objective is to maximize profits.

Production

- Production is the process of *transforming inputs into outputs*.
- Technological feasilbility determines what is possible.
- Two common ways of representing production:
- a *production possibility set* $Y \subset \mathbb{R}^n$
  - Each vector $\boldsymbol{y} \in Y$ is a *production plan* whose components are the various inputs and outputs.
  - By convention, $y_i < 0$ if resource i is used up, and $y_i > 0$ if it is produced.
  - Possible to have multiple outputs, resources that are both inputs and outputs.
- a *production function*:
  - one output, $y = f(\boldsymbol{x})$.

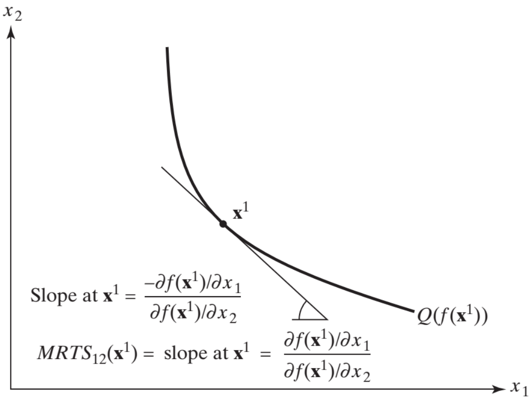
Production Functions

- We will assume that the production function $f(\boldsymbol{x})$ is continuous, strictly increasing, strictly quasiconcave, and $f(\mathbf{0}) = 0$.
- Here, the quasiconcave assumption implies there are *complementarities* in production, that is, production is decreased when there is an extreme amount of one input.
- The *marginal product of input i* is $\frac{\partial f(\boldsymbol{x})}{\partial x_i}$.
- If f is differentiable, then $\frac{\partial f(\boldsymbol{x})}{\partial x_i} > 0$.
- For a given level of output y, the set of input vectors producing y is called the y-level *isoquant*:

$$Q(y) = \{\boldsymbol{x} \geq 0 | f(\boldsymbol{x}) = y\}$$

- Simiar to indifference curves of utility functions.
- For a given input vector $\boldsymbol{x}$, the *isoquant through* $\boldsymbol{x}$ is $Q(f(\boldsymbol{x}))$, the set of input vectors producing the same output as $\boldsymbol{x}$.

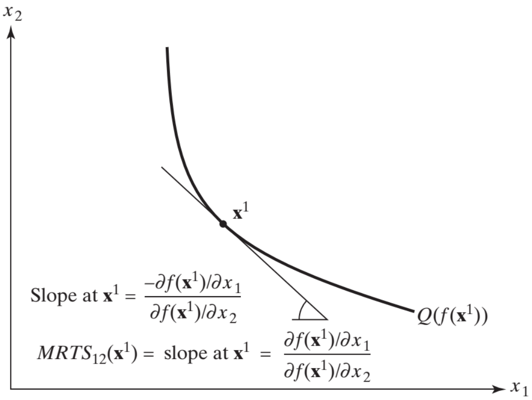
Marginal Rate of Technical Substitution



- The marginal rate of technical substitution (MRTS) is:

$$MRTS_{ij}(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})/\partial x_i}{\partial f(\boldsymbol{x})/\partial x_j}$$

Marginal Rate of Technical Substitution



- MRTS is similar to MRS for consumers: the rate at which one input can be substituted for another, while keeping output level constant.
- We will denote $\frac{\partial f(\boldsymbol{x})}{\partial x_i}$ as $f_i(\boldsymbol{x})$.

Separable Production Functions

- In general, the MRTS at inputs $\boldsymbol{x}$ depends on the level of each input.
- Frequently, we would like to divide inputs into "types". The MRTS of inputs within one type is not affected by inputs in another type.
- Suppose there are N inputs, and we can partition them into $S > 1$ mutually exclusive subsets $N_1 \ldots N_S$. The production function is *weakly separable* if the MRTS between two inputs is independent of inputs used in other groups:

$$\frac{\partial(f_i(\boldsymbol{x})/f_j(\boldsymbol{x}))}{\partial x_k} = 0 \qquad \text{for } i, j \in N_s, k \notin N_s$$

- The production function is *strongly separable* if the MRTS between two inputs from any two groups is independent of all inputs outside those two groups.

$$\frac{\partial(f_i(\boldsymbol{x})/f_j(\boldsymbol{x}))}{\partial x_k} = 0 \qquad \text{for } i \in N_s, j \in N_t, k \notin N_s \cup N_t$$

Separable Production Functions

- For example: suppose we have two types of inputs, "capital" and "labor".
- If the MRTS between two "capital" inputs is unaffected by the level of "labor" inputs, it is weakly separable.

Elasticity of Substitution

- The elasticity of substitution of input j for input i at point $\mathbf{x^0}$ is:

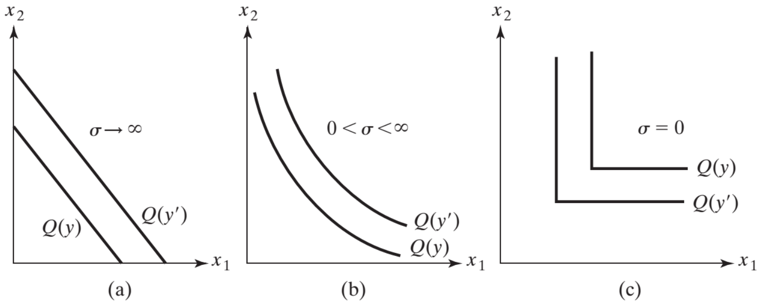
$$\sigma_{ij}(\mathbf{x^0}) = \left(\frac{d \ln MRTS_{ij}(\mathbf{x}(r))}{d \ln r} \Big|_{r=\frac{x_j^0}{x_i^0}} \right)^{-1}$$

- $r = \frac{x_j^0}{x_i^0}$, the ratio of x_j to x_i at $\mathbf{x^0}$
- $\mathbf{x}(r)$ is the vector of inputs on $Q(f(\mathbf{x}^0))$ that maintains the ratio r.
- $\sigma_{ij}(\mathbf{x^0})$ is a measure of the curvature of the isoquant through $\mathbf{x}^0$.
- If f is quasiconcave, $\sigma_{ij} \geq 0$.
- The larger σ_{ij}, the easier it is to substitute between i and j.

Derivation of σ

$$\frac{d\ln\left(\frac{x_j}{x_i}\right)}{d\ln MRTS_{ij}} = \frac{d\ln\left(\frac{x_j}{x_i}\right)}{d\ln\left(\frac{df}{dx_i}\Big/\frac{df}{dx_j}\right)}$$

$$= \frac{d(x_j/x_i)}{x_j/x_i}\frac{\frac{df}{dx_i}\Big/\frac{df}{dx_j}}{d\left(\frac{df}{dx_i}\Big/\frac{df}{dx_j}\right)}$$



- In (a), σ is infinite; there is perfect substitutability.
- In (b), $\sigma > 0$; less than perfect substitutability
- In (c), $\sigma = 0$; there is no substitutability (it is impossible to lower x_1 below a certain point, and still maintain the same output level)

Example 3.1: CES Production

- Suppose the production function is CES:

$$y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$$

$$MRTS_{12}(x_1, x_2) = \left(\frac{x_2}{x_1}\right)^{1-\rho}$$

- Set $r = \frac{x_2}{x_1}$.

$$\frac{d \ln MRTS_{12}(\boldsymbol{x}(r))}{d \ln r} = \frac{d \ln r^{1-\rho}}{d \ln r}$$

$$= (1-\rho)\frac{d \ln r}{d \ln r} = 1 - \rho$$

- Elasticity is a constant, hence CES.

CES & Cobb-Douglas

- CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$
- $\sigma = 1/(1-\rho)$, as $\rho \to 1$, σ increases, therefore x_1, x_2 become *more* substitutable for each other.
- Consider a modified version of CES with weights α_i:

$$y = \left(\sum_{i=1}^{n} \alpha_i x_i^\rho \right)^{\frac{1}{\rho}}, \sum_{i=1}^{n} \alpha_i = 1$$

$$MRTS_{ij}(x_i, x_j) = \frac{\alpha_j}{\alpha_i} \left(\frac{x_j}{x_i} \right)^{1-\rho}$$

- As $\rho \to 0, MRTS_{ij} = \frac{\alpha_j}{\alpha_i} \left(\frac{x_i}{x_j} \right)$
- This is the *MRTS* of the Cobb-Douglas production function:

$$y = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

- Therefore, Cobb-Douglas is a limiting case of CES.

CES & Leontief

- CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$
- $\sigma_{ij} = 1/(1-\rho)$ for all $i \neq j$.
- As $\rho \to -\infty, \sigma_{ij} \to 0$, there is *no* substitutability between outputs
- This is the Leontief production function:

$$y = \min(x_1, ..., x_n)$$

- Leontief production is also a limiting case of CES.

Homogeneous Production Functions are Concave

- Theorem 3.1: Let $f(\mathbf{x})$ be a continuous, strictly increasing, strictly quasiconcave production function, and suppose that f is homogeneous of degree $0 < \alpha \leq 1$. Then $f(\mathbf{x})$ is concave.
- Proof: First, suppose $\alpha = 1$. Take any strictly positive $\mathbf{x}^1, \mathbf{x}^2$, let $y^1 = f(\mathbf{x}^1), y^2 = f(\mathbf{x}^2)$.

$$f\left(\frac{\mathbf{x}^1}{y^1}\right) = f\left(\frac{\mathbf{x}^2}{y^2}\right) = 1$$

- By strict quasiconcavity of f,

$$f\left(\frac{t\mathbf{x}^1}{y^1} + \frac{(1-t)\mathbf{x}^2}{y^2}\right) \geq 1 \qquad \text{for } 0 \leq t \leq 1$$

- Choose $t^* = \frac{y^1}{y^1+y^2}, (1-t^*) = \frac{y^2}{y^1+y^2}$.

$$f\left(\frac{\mathbf{x}^1}{y^1+y^2} + \frac{\mathbf{x}^2}{y^1+y^2}\right) \geq 1$$

Homogeneous Production Functions are Concave

$$f\left(\frac{\mathbf{x}^1}{y^1+y^2}+\frac{\mathbf{x}^2}{y^1+y^2}\right)\geq 1$$

- By linear homogeneity of f,

$$f(\mathbf{x}^1+\mathbf{x}^2)\geq y^1+y^2=f(\mathbf{x}^1)+f(\mathbf{x}^2)$$

- Now, consider any $0\leq t\leq 1$, and substitute $t\mathbf{x}^1,(1-t)\mathbf{x}^2$ into the previous equation:

$$f(t\mathbf{x}^1+(1-t)\mathbf{x}^2)\geq f(t\mathbf{x}^1)+f(t\mathbf{x}^2)=tf(\mathbf{x}^1)+(1-t)f(\mathbf{x}^2)$$

- which is the characterization of a concave function.

Homogeneous Production Functions are Concave

- Now, we consider the case where $f(\mathbf{x})$ is homogeneous of degree $\alpha, 0 < \alpha \leq 1$.
- The composition $(f(\mathbf{x}))^{\frac{1}{\alpha}}$ is homogeneous of degree 1, and satisfies the assumptions on a production function (continuous, strictly increasing, strictly quasiconcave, $f(0) = 0$).
- Therefore, $(f(\mathbf{x}))^{\frac{1}{\alpha}}$ is concave, as shown above.
- The further composition $\left[(f(\mathbf{x}))^{\frac{1}{\alpha}}\right]^{\alpha} = f(\mathbf{x})$ has an outer function that is concave (since $\alpha \leq 1$).
- Therefore, the whole thing is concave, so $f(\mathbf{x})$ is concave.

Returns to Scale

- We want to know how output responds as the amounts of inputs change.
- *Returns to scale* refers to how output changes when all inputs change proportionally, i.e. multiply all inputs by the same number.
- The *marginal product* of input $i: MP_i(\mathbf{x}) = f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}$
- The *average product* of input $i: AP_i(\mathbf{x}) = f(\mathbf{x})/x_i$
- The *output elasticity* of input i, i.e. the percentage change in output resulting from a 1% change in input i:

$$\mu_i(\mathbf{x}) = \frac{f_i(\mathbf{x})x_i}{f(\mathbf{x})} = \frac{MP_i(\mathbf{x})}{AP_i(\mathbf{x})}$$

Global Returns to Scale

- A production function $f(\boldsymbol{x})$ has the following property if the corresponding condition holds for all $\boldsymbol{x}$:
  - *Constant* returns to scale if $f(t\boldsymbol{x}) = tf(\boldsymbol{x})$ for all $t > 0$.
  - *Increasing* returns to scale if $f(t\boldsymbol{x}) > tf(\boldsymbol{x})$ for all $t > 1$.
  - *Decreasing* returns to scale if $f(t\boldsymbol{x}) < tf(\boldsymbol{x})$ for all $t > 1$.
- A production function with constant returns to scale must be a linear homogeneous function.
- Every homogeneous production function with degree > 1 must have increasing returns; degree < 1 must have decreasing returns.
- The converse is not necessarily true.

Local Returns to Scale

- A production function need not have the same returns to scale everywhere.
- We would like a *local* measure of returns to scale at point $\boldsymbol{x}$.
- The *elasticity of scale* or *overall elasticity of output* is:

$$\mu(\boldsymbol{x}) = \lim_{t \to 1} \frac{d \ln [f(t\boldsymbol{x})]}{d \ln(t)} = \frac{\sum_{i=1}^{n} f_i(\boldsymbol{x}) x_i}{f(\boldsymbol{x})}$$

- Compare to the output elasticity of a single input i:

$$\mu_i(\boldsymbol{x}) = \frac{f_i(\boldsymbol{x}) x_i}{f(\boldsymbol{x})} = \frac{MP_i(\boldsymbol{x})}{AP_i(\boldsymbol{x})}$$

- If $\mu(\boldsymbol{x})$ is equal/greater than/less than 1, returns to scale are locally constant/increasing/decreasing.
- Elasticity of scale and output elasticities of input i are related:

$$\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu_i(\boldsymbol{x})$$

Example 3.2

- Consider this production function with variable returns to scale:

$$y = k(1 + x_1^{-\alpha} x_2^{-\beta})^{-1}$$

- where $\alpha > 0, \beta > 0$, and k is an upper bound on the level of output: y cannot exceed k.

$$f_1(\boldsymbol{x}) = \frac{\alpha k x_1^{\alpha-1} x_2^{\beta}}{(1 + x_1^{\alpha} x_2^{\beta})^2}, f_2(\boldsymbol{x}) = \frac{\beta k x_1^{\alpha} x_2^{\beta-1}}{(1 + x_1^{\alpha} x_2^{\beta})^2}$$

$$\mu_1(\boldsymbol{x}) = \frac{f_1(\boldsymbol{x}) x_1}{f(\boldsymbol{x})} = \frac{\alpha x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}, \mu_2(\boldsymbol{x}) = \frac{f_2(\boldsymbol{x}) x_2}{f(\boldsymbol{x})} = \frac{\beta x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}$$

$$\mu(\boldsymbol{x}) = \frac{(\alpha + \beta) x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}$$

- Elasticities depend on x_1, x_2.

Example 3.2

- We can rewrite the elasticities to be functions of the level of output $y = k(1 + x_1^{-\alpha} x_2^{-\beta})^{-1}$.

$$x_1^{-\alpha} x_2^{-\beta} = \frac{k}{y} - 1$$

$$\mu_1^*(y) = \frac{\alpha x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}} = \alpha \left(1 - \frac{y}{k}\right)$$

$$\mu_2^*(y) = \frac{\beta x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}} = \beta \left(1 - \frac{y}{k}\right)$$

$$\mu^*(y) = (\alpha + \beta) \left(1 - \frac{y}{k}\right)$$

- Returns to scale decline monotonically as output increases.
- At $y = 0, u^*(y) = \alpha + \beta > 0$. At $y \to k, u * (y) \to 0$.
- If $\alpha + \beta > 1$, returns to scale will go from increasing to decreasing as y goes up.

Cost Function

- The firm's *cost of output* of a level y is the expenditure it must make to acquire the inputs necessary to produce y.
- If the firm's objective is to maximize profits, then for a given y, it must try to minimize costs (since revenues are determined by y).
- We will assume that firms are perfectly competitive on their input markets, and therefore are *price-takers*.
- Let $\boldsymbol{w} = (w_1, \ldots, w_n)$ denote the market prices of inputs $1, \ldots n$.
- Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ denote the quantities of inputs $1, \ldots n$.
- The firm will choose $\boldsymbol{x}$ to minimize the expenditure $\boldsymbol{w} \cdot \boldsymbol{x}$, subject to producing output $f(\boldsymbol{x}) \geq y$.

Cost Function

- The *cost function*, for strictly positive input prices $\boldsymbol{w}$ and feasible output levels $y \in f(\mathbb{R}^n_+)$, is defined as:

$$c(\boldsymbol{w}, y) = \min_{\boldsymbol{x} \in \mathbb{R}^n_+} \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad f(\boldsymbol{x}) \geq y$$

- Let $\boldsymbol{x}(\boldsymbol{w}, y)$ denote the solution to this problem. Then:

$$c(\boldsymbol{w}, y) = \boldsymbol{w} \cdot \boldsymbol{x}(\boldsymbol{w}, y)$$

- Since we assume that $f(\cdot)$ is strictly increasing, the constraint will always be binding at a solution (similar to: monotonicity of utility guarantees the budget constraint is satisfied with equality).

$$c(\boldsymbol{w}, y) = \min_{\boldsymbol{x} \in \mathbb{R}^n_+} \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad f(\boldsymbol{x}) = y$$

Cost Function

$$c(\boldsymbol{w}, y) = \min_{\boldsymbol{x} \in \mathbb{R}^n_+} \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad f(\boldsymbol{x}) = y$$

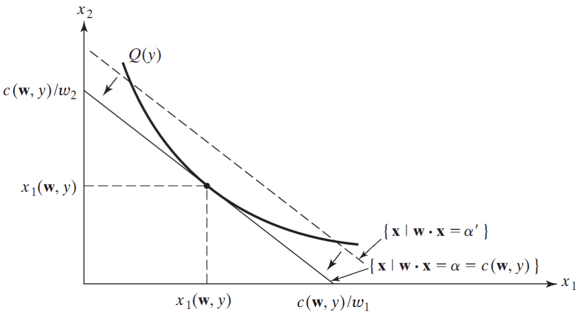
$$L(x_1, ..., x_n, \lambda) = \boldsymbol{w} \cdot \boldsymbol{x} - \lambda(f(\boldsymbol{x}) - y)$$

$$\frac{\partial L}{\partial x_i} = w_i - \lambda \frac{\partial f(\boldsymbol{x})}{\partial x_i}$$

$$w_i = \lambda^* \frac{\partial f(\boldsymbol{x})}{\partial x_i} \qquad \text{for } i = 1, ..., n$$

$$MRTS_{i,j} = \frac{\partial f(\boldsymbol{x^*})/\partial x_i}{\partial f(\boldsymbol{x^*})/\partial x_j} = \frac{w_i}{w_j}$$

Conditional Input Demand



- The solution $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{w}, y)$ is called the firm's *conditional input demand*, i.e. demand conditional on producing at least output level y.
- The solution to the cost-minimization problem is the tangency point between the y-level isoquant, and an *isocost* line of the form $\boldsymbol{w} \cdot \boldsymbol{x} = \alpha$ for some $\alpha > 0$.

Example 3.3

- Suppose production is CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$
- Cost minimization problem:

$$\min_{x_1 \geq 0, x_2 \geq 0} w_1 x_1 + w_2 x_2 \qquad \text{s.t.} \qquad (x_1^\rho + x_2^\rho) = y$$

- First-order conditions:

$$MRTS_{12} = \frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1}$$

$$y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = x_2 w_2^{\frac{-1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}$$

- Conditional input demands:

$$x_1 = y w_1^{\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}}$$

$$x_2 = y w_2^{\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}}$$

Example 3.3

$$x_1 = y w_1^{\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}}$$

$$x_2 = y w_2^{\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \frac{1}{\rho} \right)^{\frac{-1}{\rho}}$$

- Plug x_1^*, x_2^* into $w_1 x_1 + w_2 x_2$ to get cost function:

$$c(w_1, w_2, y) = y \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}$$

Similarities to Expenditure Function

- Obviously, this is very similar to the expenditure function in consumer theory.
- Expenditure function:

$$e(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad u(\boldsymbol{x}) \geq u$$

- Cost function:

$$c(\boldsymbol{w}, u) = \min_{\boldsymbol{x}} \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad f(\boldsymbol{x}) \geq u$$

- All the properties and theorems that apply to expenditure functions also apply to cost functions.

Properties of Cost Function

- Theorem 3.2: If f is continuous and strictly increasing, then $c(\boldsymbol{w}, y)$ has the following properties:
  - $c(\boldsymbol{w}, 0) = 0$
  - is continuous on its domain
  - For strictly positive $\boldsymbol{w}$, is strictly increasing and unbounded above in y
  - Increasing in $\boldsymbol{w}$
  - Homogeneous of degree 1 in $\boldsymbol{w}$
  - Concave in $\boldsymbol{w}$
- If f is strictly quasiconcave (therefore guaranteeing a unique solution), then

$$\frac{\partial c(\boldsymbol{w}^{\mathbf{0}}, y^0)}{\partial w_i} = x_i(w^0, y^0) \qquad \text{for } i = 1, \dots, n$$

- The last property is known as Shephard's lemma.

Example 3.4

- Suppose the cost function is Cobb-Douglas: $c(w_1, w_2, y) = Aw_1^{\alpha} w_2^{\beta} y$
- By Shephard's lemma, conditional input demand is $\frac{\partial c}{\partial w_i}$:

$$x_1(w_1, w_2, y) = \alpha A w_1^{\alpha-1} w_2^{\beta} y = \frac{\alpha c(w_1, w_2, y)}{w_1}$$

$$x_2(w_1, w_2, y) = \beta A w_1^{\alpha} w_2^{\beta-1} y = \frac{\beta c(w_1, w_2, y)}{w_2}$$

- Ratio of conditional input demands:

$$\frac{x_1(w_1, w_2, y)}{x_2(w_1, w_2, y)} = \frac{\alpha}{\beta} \frac{w_2}{w_1}$$

- With a Cobb-Douglas cost function, the ratio of inputs depends only on the ratio of input prices, independent of the level of output.

Example 3.4

- Define the *input share* of input i as: $s_i = \frac{w_i x_i(w_1, w_2, y)}{c(w_1, w_2, y)}$, i.e. the fraction of total expenditure spent on input i.
- For this cost function, $s_1 = \alpha, s_2 = \beta$.

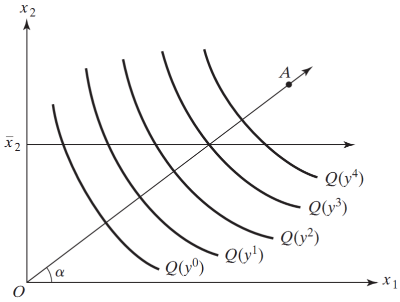
Properties of Conditional Input Demands

- These are the same as the properties of Hicksian compensated demands.
- Theorem 3.3: Suppose f is continuous, strictly increasing, and strictly quasiconcave. Then:
  - $\boldsymbol{x}(\boldsymbol{w}, y)$ is homogeneous of degree zero in $\boldsymbol{w}$
  - The substitution matrix $\sigma^*(\boldsymbol{w}, y)$ is symmetric and negative semidefinite:

$$\sigma^*(\boldsymbol{w}, y) = \begin{pmatrix} \frac{\partial x_1(\boldsymbol{w},y)}{\partial w_1} & \cdots & \frac{\partial x_1(\boldsymbol{w},y)}{\partial w_n} \\ \vdots & & \vdots \\ \frac{\partial x_n(\boldsymbol{w},y)}{\partial w_1} & \cdots & \frac{\partial x_n(\boldsymbol{w},y)}{\partial w_n} \end{pmatrix}$$

  - This implies that $\frac{\partial x_i(\boldsymbol{w},y)}{\partial w_i} \leq 0$ for all i. (Recall: Hicksian demand always decreases as own price increases).

Homothetic Production Functions



- A *homothetic* production function has the property that $MRTS_{ij}(x_i, x_j)$ is homogeneous of degree zero.
- For example, with CES production, $MRTS_{12}(x_1, x_2) = \left(\frac{x_2}{x_1}\right)^{1-\rho}$.
- Along rays from the origin, the slope of the isoquants are the same.

Homothetic Production Functions

- A homothetic function can be written in the form $H(f(\mathbf{x}))$, where $H(\cdot)$ is a monotonically increasing function, and $f(\mathbf{x})$ is a homogeneous function of any degree.
- Theorem 3.4: When $f(\cdot)$ is continuous, strictly increasing, strictly quasiconcave, and homothetic, then:
  - The cost function is *multiplicatively separable* in $\mathbf{w}, y$: there is some strictly increasing $h(y)$ such that:

    $$c(\mathbf{w}, y) = h(y)c(\mathbf{w}, 1)$$

  - Conditional input demands are multiplicatvely separable in $\mathbf{w}, y$: there is some strictly increasing $h'(y)$ such that:

    $$\mathbf{x}(\mathbf{w}, y) = h'(y), \mathbf{x}(\mathbf{w}, 1)$$

Homothetic Production Functions

- When $f(\cdot)$ is homogeneous of degree $\alpha > 0$:

$$c(\mathbf{w}, y) = y^{\frac{1}{\alpha}} c(\mathbf{w}, 1)$$

$$\mathbf{x}(\mathbf{w}, y) = y^{\frac{1}{\alpha}} \mathbf{x}(\mathbf{w}, 1)$$

Short-Run Costs

- So far, we have been assuming that all inputs can be changed when calculating the appropriate cost.
- This may not be true in all situations. In particular, in the *short run*, some inputs may be fixed (e.g. if "capital" inputs, such as factories or machines, take a long time to build).
- We denote the *long-run* cost function as the one where all inputs can be changed.
- The *short-run* cost function is where some inputs are fixed.

Short-Run Costs

- Let the production function be $f(\boldsymbol{z})$, where we divide the inputs $\boldsymbol{z}$ into *variable* inputs $\boldsymbol{x}$, and *fixed* inputs $\overline{\boldsymbol{x}}$:

$$\boldsymbol{z} = (\boldsymbol{x}, \overline{\boldsymbol{x}})$$

- The *short-run*, or *restricted* cost function, is defined as:

$$sc(\boldsymbol{w}, \overline{\boldsymbol{w}}, y; \overline{\boldsymbol{x}}) = \min_{\boldsymbol{x}} \boldsymbol{w} \cdot \boldsymbol{x} + \overline{\boldsymbol{w}} \cdot \overline{\boldsymbol{x}} \qquad \text{s.t. } f(\boldsymbol{x}, \overline{\boldsymbol{x}}) \geq y$$

- That is, we are taking the fixed inputs $\overline{\boldsymbol{x}}$ as parameters to the problem, but they still contribute to expenditures.
- If $\boldsymbol{x}(\boldsymbol{w}, \overline{\boldsymbol{w}}, y; \overline{\boldsymbol{x}})$ solves this problem, then

$$sc(\boldsymbol{w}, \overline{\boldsymbol{w}}, y; \overline{\boldsymbol{x}}) = \boldsymbol{w} \cdot \boldsymbol{x}(\boldsymbol{w}, \overline{\boldsymbol{w}}, y; \overline{\boldsymbol{x}}) + \overline{\boldsymbol{w}} \cdot \overline{\boldsymbol{x}}$$

Short-Run Costs

- The optimized cost of the variable inputs, $\boldsymbol{w} \cdot \boldsymbol{x}(\boldsymbol{w}, \overline{\boldsymbol{w}}, y; \overline{\boldsymbol{x}})$, is called *total variable cost*.
- The cost of the fixed inputs, $\overline{\boldsymbol{w}} \cdot \overline{\boldsymbol{x}}$, is called *total fixed cost*.

Properties of Short-Run Costs

- For a given level of output, long-run costs (where the firm can set all input levels) cannot be greater than short-run costs (where the firm cannot set all input levels).
- This is because the feasible region for $\boldsymbol{z}$ of the short-run problem is a *subset* of the feasible region of the long-run problem.
- Any feasible $\boldsymbol{z}$ in the short-run problem is feasible in the long-run problem, but not vice versa.
- In general, if set $A \subset B$, then for any function g:

$$\max_{x \in A} g(x) \leq \max_{x \in B} g(x)$$

$$\min_{x \in A} g(x) \geq \min_{x \in B} g(x)$$

Duality

- In consumer theory, we saw the duality between utility and expenditure.
- Likewise, there is a duality between production and cost.
- If we begin with a production function, we can derive the cost function.
- If we begin with a cost function, we can generate a production function. If the original production function is quasiconcave, the derived production function will be identical.
- This is important for applied work. When analyzing real-world data, it is difficult to know what the "true" production function is.
- Instead, we can observe market input prices and levels of output, and estimate a cost function.
- Then, "recover" the underlying production function.

Recovering Production Function from Cost Function

- Theorem 3.5: Suppose $c(\boldsymbol{w}, y)$ satisfies the properties of a cost function (including differentiability). Then the function $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ defined by:

$$f(\boldsymbol{x}) = max\{y \geq 0 | \boldsymbol{w} \cdot \boldsymbol{x} \geq c(\boldsymbol{w}, y) \qquad \forall \boldsymbol{x} >> 0\}$$

- is increasing, unbounded above, and quasiconcave. Moreover, the cost function generated by f is c.

The Competitive Firm

- Now, let's assume the firm is in perfect competition on both input and output markets, i.e. it is a price taker for both input and output prices.
- Let p denote the output market price.
- Revenues are: $R(y) = py$
- For a given output level y, profits are: $py - c(\mathbf{w}, y)$
- We will consider the problem where the firm can choose both the level of output, and the inputs used to produce it.

Profit Maximization Problem

$$\max_{\boldsymbol{x},y} py - \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t.} \qquad f(\boldsymbol{x}) \geq y$$

- Assuming f is strictly increasing, the constraint will be satisfied with equality. Therefore, y is completely determined by $\boldsymbol{x}$, so the problem becomes:

$$\max_{\boldsymbol{x}} pf(\boldsymbol{x}) - \boldsymbol{w} \cdot \boldsymbol{x}$$

- Assume this problem has an interior solution at some strictly positive $\boldsymbol{x}^*$.

Profit Maximization Problem

- First-order conditions require that the gradient be zero, since there are no constraints:

$$p\frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} = w_i \qquad \text{for } i = 1, ..., n$$

- The term on the left-hand side is called the *marginal revenue product* of input i.
- At optimality, marginal revenue product = marginal cost of input i.

Profit Maximization Problem

- Assuming all input prices w_i are positive, we get that the MRTS must equal the ratio of prices:

$$MRTS_{ij} = \frac{\partial f(\boldsymbol{x}^*)/\partial x_i}{\partial f(\boldsymbol{x}^*)/\partial x_j} = \frac{w_i}{w_j} \qquad \text{for all } i, j$$

- This is the same condition as in the cost-minimization problem.
- Input demands will be the same as in the cost-minimization problem.

Profit Function

- Suppose f satisfies the usual conditions, and in addition, is *strictly concave*.
- Then, solutions to the profit maximization problem will be unique for each $(p, \boldsymbol{w})$.
- The optimal choice of output, $y^* = y(p, \boldsymbol{w})$ is called the firm's *output supply function*.
- The optimal choice of inputs, $\boldsymbol{x}^* = \boldsymbol{x}(p, \boldsymbol{w})$ is called the firm's *input demand function*.
- The *profit function* depends only on input and output prices:

$$\pi(p, \boldsymbol{w}) = \max_{\boldsymbol{x}, y} py - \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t. } f(\boldsymbol{x}) \geq y$$

Profit Function

$$\pi(p, \boldsymbol{w}) = \max_{\boldsymbol{x}, y} py - \boldsymbol{w} \cdot \boldsymbol{x} \qquad \text{s.t. } f(\boldsymbol{x}) \geq y$$

- First, we must be sure that a maximum of profits actually exists (i.e. is finite).
- It may be that there is no finite maximum. For example, suppose that production has increasing returns.
- Then, starting from any choice of $\boldsymbol{x}$ and $y = f(\boldsymbol{x})$, it will always be possible to choose $t\boldsymbol{x}$, $t > 0$, that gives higher profits.
- If production has constant returns, then a maximum profit may exist. However, the *scale* is indeterminate: the input levels $\boldsymbol{x}$ and $t\boldsymbol{x}$ give the same profits for all $t > 0$.

Properties of the Profit Function

- Theorem 3.7: Assume f is continuous, strictly increasing, and strictly quasiconcave. For $p \geq 0, \boldsymbol{w} \geq 0$, the profit function $\pi(p, \boldsymbol{w})$ (where well-defined) is continuous and:
  - Increasing in p;
  - Decreasing in $\boldsymbol{w}$;
  - Homogeneous of degree one in $(p, \boldsymbol{w})$;
  - Convex in $(p, \boldsymbol{w})$;
  - Differentiable in $(p, \boldsymbol{w})$ for strictly positive $(p, \boldsymbol{w})$. If f is strictly concave, then

$$\frac{\partial \pi(p, \boldsymbol{w})}{\partial p} = y(p, \boldsymbol{w}), -\frac{\partial \pi(p, \boldsymbol{w})}{\partial w_i} = x_i(p, \boldsymbol{w}) \qquad \text{for } i = 1, \dots, n$$

- The last property is known as Hotelling's Lemma.

Properties of Output Supply and Input Demand Functions

- Theorem 3.8: Assume f is continuous, strictly increasing, and strictly concave, and assume $\pi(p, \mathbf{w})$ is twice continuously differentiable. Then, for all $p > 0, \mathbf{w} >> 0$ where it is well-defined:
  - Homogeneity of degree zero:

$$y(tp, t\mathbf{w}) = y(p, \mathbf{w}) \qquad \text{for all } t > 0$$

$$x_i(tp, t\mathbf{w}) = x_i(p, \mathbf{w}) \qquad \text{for all } t > 0, i = 1, ..., n$$

  - Own-price effects:

$$\frac{\partial y(p, \mathbf{w})}{\partial p} \geq 0, \frac{\partial x_i(p, \mathbf{w})}{\partial w_i} \leq 0 \qquad \text{for } i = 1, ..., n$$

Properties of Output Supply and Input Demand Functions

- Substitution matrix is symmetric and positive semidefinite:

$$\begin{pmatrix} \frac{\partial y(p,\mathbf{w})}{\partial p} & \frac{\partial y(p,\mathbf{w})}{\partial w_1} & \cdots & \frac{\partial y(p,\mathbf{w})}{\partial w_n} \\ \frac{-\partial x_1(p,\mathbf{w})}{\partial p} & \frac{-\partial x_1(p,\mathbf{w})}{\partial w_1} & \cdots & \frac{-\partial x_1(p,\mathbf{w})}{\partial w_n} \\ \vdots & \vdots & & \vdots \\ \frac{-\partial x_n(p,\mathbf{w})}{\partial p} & \frac{-\partial x_n(p,\mathbf{w})}{\partial w_1} & \cdots & \frac{-\partial x_n(p,\mathbf{w})}{\partial w_n} \end{pmatrix}$$

Example 3.5

- Assume CES production: $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$.
- Note the parameter β. This determines the scale of production; if $\beta < 1$, there is decreasing returns to scale. Suppose $\beta < 1$.
- Assume an interior solution. First-order conditions:

$$w_1 + p\beta(x_1^\rho + x_2^\rho)^{\frac{\beta-\rho}{\rho}} x_1^{\rho-1} = 0$$

$$w_2 + p\beta(x_1^\rho + x_2^\rho)^{\frac{\beta-\rho}{\rho}} x_2^{\rho-1} = 0$$

$$x_1 = x_2(w_1/w_2)^{\frac{1}{\rho-1}}$$

- The supply function is:

$$y = (p\beta)^{\frac{-\beta}{\beta-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\beta(\rho-1)}{\rho(\beta-1)}}$$

Example 3.5

- The input demand functions are:

$$x_i = w_i^{\frac{1}{\rho-1}} (p\beta)^{\frac{-1}{\beta-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-\beta}{\rho(\beta-1)}}$$

- The profit function is:

$$\pi(p, \boldsymbol{w}) = p^{\frac{-1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \beta^{\frac{-\beta}{\beta-1}} (1-\beta), \qquad r = \frac{\rho}{\rho-1}$$

- If $\beta = 1$, production has constant returns to scale and the profit function is undefined.
- If $\beta > 1$, production has increasing returns to scale. The FOC give conditions for a minimum instead of a maximum.

Short-Run Profit Maximization

- As before, the long-run profit function is when all inputs can be changed. The short-run profit function is when some inputs must be fixed.
- Theorem 3.9: Suppose that f is continuous, strictly increasing, and strictly concave.
  - For $k < n$, let $\overline{\boldsymbol{x}} \in \mathbb{R}_+^k$ be a subvector of fixed inputs
  - Consider $f(\boldsymbol{x}, \overline{\boldsymbol{x}})$ as a function of the subvector of variable inputs $\boldsymbol{x} \in \mathbb{R}_+^{n-k}$.
  - Let $\boldsymbol{w}, \overline{\boldsymbol{w}}$ be the vector of prices for the variable and fixed inputs, respectively.
  - The *short-run*, or *restricted*, profit function is:

$$\pi(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}}) = \max_{y, \boldsymbol{x}} py - \boldsymbol{w} \cdot \boldsymbol{x} - \overline{\boldsymbol{w}} \cdot \overline{\boldsymbol{x}} \qquad \text{s.t. } f(\boldsymbol{x}, \overline{\boldsymbol{x}}) \geq y$$

- The solutions $y(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}}), \boldsymbol{x}(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}})$ are called the short-run output supply and variable input demand functions.

Short-Run (or Restricted) Profit Function

- For all $p > 0$ and $\boldsymbol{w} >> 0, \pi(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}})$, where well-defined, is:
  - continuous in p and $\boldsymbol{w}$,
  - increasing in p
  - decreasing in $\boldsymbol{w}$
  - convex in $(p, \boldsymbol{w})$
  - If $\pi(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}})$ is twice continuously differentiable, the short-run output supply and variable input demand functions have the same properties as in Theorem 3.8 (homogeneity of degree zero, own-price effects, and positive semidefinite substitution matrix)

Optimal Shutdown

- Recall that $sc(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}})$ is the short-run cost function. Consider the short-run profit function

$$\pi(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}}) = max_y py - sc(p, \boldsymbol{w}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{x}})$$

- The first-order condition tells us that for optimal $y^* > 0$,

$$p = \frac{d \ sc(y^*)}{dy}$$

- that is, price should equal short-run marginal cost. Suppose this is true at some y^1.
- Let $tvc(y)$ denote the total variable cost, and let tfc denote the total fixed cost. Then

$$\pi^1 = py^1 - tvc(y^1) - tfc$$

Optimal Shutdown

- If π^1 is negative, the firm is better off shutting down and producing nothing ($y = 0$). Let π^0 denote profits when $y = 0$:

$$\pi^0 = p \cdot 0 - tvc(0) - tfc = -tfc < 0$$

- The firm will produce $y^1 > 0$ only if $\pi^1 \geq \pi^0$, or

$$py^1 - tvc(y^1) \geq 0$$

$$p \geq \frac{tvc(y^1)}{y^1} = avc(y^1)$$

- Thus, the firm will shut down if the output price p is less than the average variable cost of y^1.

Homework #2

- Homework #2 is due next meeting.
- The midterm will be on April 17.
- Midterm will be open-book.
- Chapters 1, 2.1, 2.4, 3, and possibly 4 will be covered.
- Sample midterms from previous semesters are on the course website.