**Advanced Microeconomic Analysis**

**Solutions to Homework #1**

## 0.1 A2.9

$f(x_1, x_2) = (x_1 x_2)^2, g(x_1, x_2) = (x_1^2 x_2)^3$.

(a) $f(tx_1, tx_2) = (tx_1 \cdot tx_2)^2 = t^4(x_1 x_2)^2$, so homogeneous of degree 4.

(b) $g(tx_1, tx_2) = ((tx_1)^2 tx_2)^3 = (t^3 x_1^2 x_2)^3 = t^9(x_1^2 x_2)^3$, so homogeneous of degree 9.

(c) $f(tx_1, tx_2)g(tx_1, tx_2) = t^1 3(x_1 x_2)^2(x_1^2 x_2)^3$, so homogeneous of degree 13.

(d) $k(tx_1, tx_2) = g(f(tx_1, tx_2), f(tx_1, tx_2)) = g(t^4 f(tx_1, tx_2), t^4 f(tx_1, tx_2)) = (t^4)^9 g(f(x_1, x_2), f(x_1, x_2)) = t^3 6k(tx_1, tx_2)$, so homogeneous of degree 36.

(e) Suppose $f(x_1, x_2)$ is HOD-$m$ and $g(x_1, x_2)$ is HOD-$n$. $k(tx_1, tx_2) = g(f(tx_1, tx_2), f(tx_1, tx_2)) = g(t^m f(x_1, x_2), t^m f(x_1, x_2)) = (t^m)^n g(f(x_1, x_2), f(x_1, x_2)) = t^{mn} k(x_1, x_2)$

## 0.2 A2.25

For each problem, we form the Lagrangian function, write down the first-order conditions, and solve for the optimal values.

(d) Objective function: $f(x_1, x_2) = x_1 + x_2$. Constraint: $g(x_1, x_2) = x_1^4 + x_2^4 - 1 = 0$.

$$\max_{x_1, x_2} x_1 + x_2 \qquad \text{s.t. } x_1^4 + x_2^4 - 1 = 0$$

$$L(x_1, x_2, \lambda) = x_1 + x_2 - \lambda(x_1^4 + x_2^4 - 1)$$

$$\frac{\partial L}{\partial x_1} = x_1 - \lambda 4x_1^3 = 0 \Rightarrow x_1 = \lambda 4x_1^3$$

$$\frac{\partial L}{\partial x_2} = x_2 - \lambda 4x_2^3 = 0 \Rightarrow x_2 = \lambda 4x_2^3$$

$$\frac{\partial L}{\partial \lambda} = x_1^4 + x_2^4 - 1 = 0$$

$$\frac{x_1}{x_2} = \frac{x_1^3}{x_2^3} \Rightarrow x_1 = x_2$$

Plugging into constraint,

$$x_1^4 + x_1^4 = 1 \Rightarrow x_1^* = 2^{-\frac{1}{4}}, x_2^* = 2^{-\frac{1}{4}}, \lambda^* = 2^{-\frac{3}{2}}$$

The maximized value of the objective function is:

$$x_1^* + x_2^* = 2^{\frac{3}{4}}$$

(e) Objective function: $f(x_1, x_2) = x_1 x_2^2 x_3^3$. Constraint: $g(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 1 = 0$.

$$\max_{x_1, x_2, x_3} x_1 x_2^2 x_3^3 \qquad \text{s.t. } x_1 + x_2 + x_3 - 1 = 0$$

$$L(x_1, x_2, x_3, \lambda) = x_1 x_2^2 x_3^3 - \lambda(x_1 + x_2 + x_3 - 1)$$
$$\frac{\partial L}{\partial x_1} = x_2^2 x_3^3 - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_1 x_2 x_3^3 - \lambda = 0$$
$$\frac{\partial L}{\partial x_3} = 3x_1 x_2^2 x_3^3 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 1 = 0$$
$$x_2^2 x_3^3 = 2x_1 x_2 x_3^3 \Rightarrow x_2 = 2x_1$$
$$2x_1 x_2 x_3^3 = 3x_1 x_2^2 x_3^3 \Rightarrow x_3 = \frac{3}{2} x_2 = 3x_1$$

Plugging into constraint,

$$x_1 + 2x_1 + 3x_1 - 1 = 0 \Rightarrow x_1^* = \frac{1}{6}, x_2^* = \frac{1}{3}, x_3^* = \frac{1}{2}$$

The maximized value of the objective function is:

$$x_1^* x_2^{*2} x_3^{*3} = \frac{1}{432}$$

## 0.3 A2.33

The maximization problem can be written as:

$$\max_{x_1 \dots x_n} f(x_1, ..., x_n) \qquad \text{subject to}$$

$$g^1(x_1, ..., x_n) - a_1 = 0$$
$$g^3(x_1, ..., x_n) - a_2 = 0$$
$$\vdots$$
$$g^m(x_1, ..., x_n) - a_m = 0$$

Denote the $j$th constraint as

$$G^j(x_1, ..., x_n, a_1, ...a_m) = g^j(x_1, ..., x_n) - a_j = 0$$

The Lagrangian is

$$L(x_1, ..., x_n, \lambda_1, ..., \lambda_m) = f(x_1, ..., x_n) - \sum_{i=1}^{m} \lambda_i G^i(x_1, ..., x_n, a_1, ...a_m)$$

By the Envelope Theorem,

$$\frac{\partial V(a_1,...a_j)}{\partial a_j} = \frac{\partial L}{\partial a_j}$$
$$= \frac{\partial f}{\partial a_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial G^i}{\partial a_j}$$

Since $a_j$ does not enter into $f(\cdot)$, the first term is 0. Each $\frac{\partial G^i}{\partial a_j}$ term is 0 if $i \neq j$, and -1 if $i = j$. Therefore, $\frac{\partial V(a_1,...a_j)}{\partial a_j} = \lambda_j$.

### 0.4 1.3

(a) For a binary relation $>$ to be complete, for all $x^1, x^2$ in $X$, one or both of these must be true: $x^1 > x^2$, $x^2 > x^1$. We assume the preference relation $\succsim$ is complete, therefore $x^1 \succsim x^2$ is true, $x^2 \succsim x^1$ is true, or both are true.

Suppose $\succ$ is complete. Then at least one of these is true: $x \succ x$, $x \prec x$. However, both are false, so $\succ$ cannot be complete.

Suppose $\sim$ is complete, and consider $x^1, x^2$ such that $x^1 \succ x^2$. Then at least one of these is true: $x^1 \sim x^2$, $x^2 \sim x^1$. However, both are false, so $\sim$ cannot be complete.

(b) By the completeness of $\succsim$, $x^1 \succsim x^2$ is true, $x^2 \succsim x^1$ is true, or both are.

- Suppose both are true. Then $x^1 \sim x^2$.
- Suppose $x^1 \succsim x^2$ is true and $x^2 \succsim x^1$ is false. Then $x^1 \succ x^2$.
- Suppose $x^1 \succsim x^2$ is false and $x^2 \succsim x^1$ is true. Then $x^2 \succ x^1$.

Since these are the only three possibilities, exactly one must hold.

### 0.5 1.5

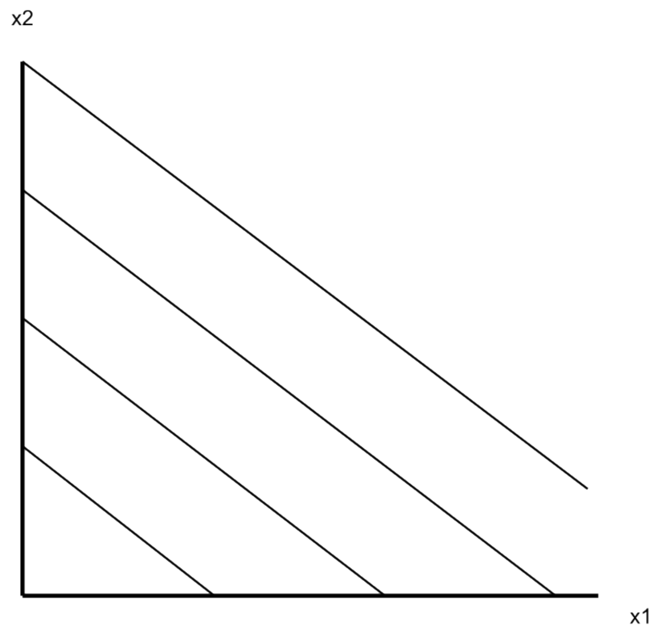
(a) By definition, $\sim (x^0) = \{x | x \sim x^0\}$. Suppose $x \in \sim (x^0)$. Then $x \sim x^0 \Rightarrow x \succsim x^0$ and $x^0 \succsim x$, therefore $x \in \{x | x \succsim x^0\} \cap \{x | x \precsim x^0\}$, or $x \in \succsim (x^0) \cap \precsim (x^0)$.

(d) Suppose $x \in \sim (x^0) = \{x | x \sim x^0\}$, therefore $x \sim x^0$. Then $x \prec x^0$ is false, therefore $x$ cannot be in $\prec (x^0) = \{x | x \prec x^0\}$.

Suppose $x \in \prec (x^0)$, therefore $x \prec x^0$. Then $x \sim x^0$ is false, therefore $x$ cannot be in $\sim (x^0)$.

Combining both statements, the intersection of $\sim (x^0)$ and $\prec (x^0)$ is empty.

## 0.6 1.8




One possible set of indifference curves is shown above. The upper level sets are convex, but not strictly convex, therefore these preferences are quasiconcave, but not strictly quasiconcave.

## 0.7 1.12

Suppose $u(x_1, x_2)$ and $v(x_1, x_2)$ are utility functions.

(a) $s(tx_1, tx_2) = u(tx_1, tx_2) + v(tx_1, tx_2) = t^r u(x_1, x_2) + t^r v(x_1, x_2) = t^r(u(x_1, x_2) + v(x_1, x_2)) = t^r s(x_1, x_2)$

(b) We will show that the level sets of $m(x_1, x_2) = \min\left[u(x_1, x_2), v(x_1, x_2)\right]$ are convex. For a given utility level $\overline{u}$, the level set of $m$ is $\{(x_1, x_2) | m(x_1, x_2) \geq \overline{u}\} = \{(x_1, x_2) | u(x_1, x_2) \geq \overline{u}$ and $v(x_1, x_2) \geq \overline{u}\}$. Therefore, for a given utility level $\overline{u}$, the level set of $m$ is the intersection of the level sets of $u$ and $v$. By assumption, $u$ and $v$ are quasiconcave, therefore their level sets are convex. The intersection of convex sets is also convex, so $m$ is also quasiconcave.

## 0.8 1.20

Suppose $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}, 0 \leq \alpha \leq 1, A > 0$. The utility maximization problem is:

$$\max_{x_1, x_2} Ax_1^\alpha x_2^{1-\alpha} \qquad \text{s.t.} \qquad p_1 x_1 + p_2 x_2 - y = 0$$

$$L(x_1, x_2, \lambda) = Ax_1^\alpha x_2^{1-\alpha} + \lambda(p_1 x_1 + p_2 x_2 - y)$$

$$\frac{\partial L}{\partial x_1} = A\alpha x_1^{\alpha-1}x_2^{1-\alpha} - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = A(1-\alpha)x_1^{\alpha}x_2^{-\alpha} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 - y = 0$$

Combining $\frac{\partial L}{\partial x_1}$ and $\frac{\partial L}{\partial x_2}$, we get

$$\frac{\alpha}{1-\alpha}\frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2}\frac{1-\alpha}{\alpha}x_1$$

Plugging into budget constraint,

$$p_1x_1 + p_2\frac{p_1}{p_2}\frac{1-\alpha}{\alpha}x_1 = y \Rightarrow x_1^* = \frac{\alpha y}{p_1}, x_2^* = \frac{(1-\alpha)y}{p_2}$$

## 0.9 1.21

Suppose $u(x_1, x_2) = \log(Ax_1^{\alpha}x_2^{1-\alpha}) = \log(A) + \alpha\log(x_1) + (1-\alpha)\log(x_2), 0 \leq \alpha \leq 1, A > 0$. The utility maximization problem is:

$$\max_{x_1,x_2} \log(A) + \alpha\log(x_1) + (1-\alpha)\log(x_2) \qquad \text{s.t.} \qquad p_1x_1 + p_2x_2 - y = 0$$

$$L(x_1, x_2, \lambda) = \log(A) + \alpha\log(x_1) + (1-\alpha)\log(x_2) - \lambda(p_1x_1 + p_2x_2 - y)$$
$$\frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = \frac{\alpha}{x_2} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 - y = 0$$

Combining $\frac{\partial L}{\partial x_1}$ and $\frac{\partial L}{\partial x_2}$, we get

$$\frac{\alpha}{1-\alpha}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

which is the same as in problem 1.20. Therefore, the generated Marshallian demand will be the same.

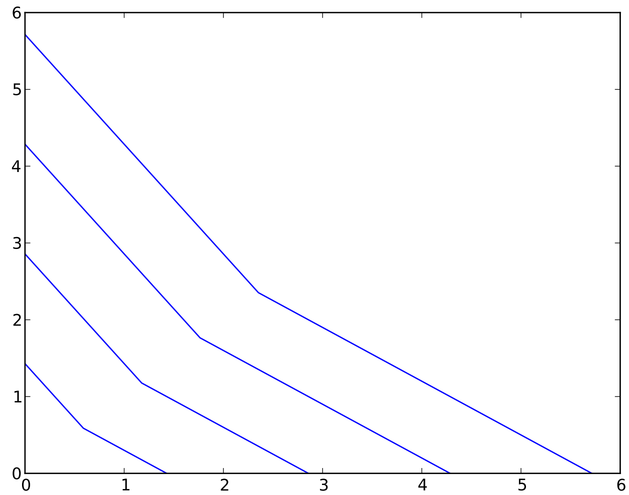
## 0.10 1.27

Suppose $u(x_1, x_2) = \max(ax_1, ax_2) + \min(x_1, x_2) = a\max(x_1, x_2) + \min(x_1, x_2), 0 \leq a \leq 1$. This is not a differentiable function, so we can't use calculus methods to solve this problem. Let's examine the shape of the indifference curves, given a utility level $\overline{u}$.

- If $x_1 > x_2$, then $u(x_1, x_2) = ax_1 + x_2 = \overline{u} \Rightarrow x_2 = \overline{u} - ax_1$
- If $x_1 < x_2$, then $u(x_1, x_2) = x_1 + ax_2 = \overline{u} \Rightarrow x_2 = \frac{\overline{u}}{a} - \frac{x_1}{a}$

- If $x_1 = x_2$, then $u(x_1, x_2) = (1+a)x_1 = (1+a)x_2 = \overline{u}$.

The indifference curves for $a = 0.7$ are shown below:




The optimal solution(s) will depend on the slope of the budget line, $-\frac{p_1}{p_2}$.

- If $-\frac{p_1}{p_2} < -\frac{1}{a}$, the optimal solution will be the corner solution at $(0, y/p_2)$.
- If $-\frac{p_1}{p_2} = -\frac{1}{a}$, all points on the upper half of the indifference curve are optimal.
- If $-a > -\frac{p_1}{p_2} > -\frac{1}{a}$, the only optimal solution is the point on the 45-degree line, $(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2})$.
- If $-a = -\frac{p_1}{p_2}$, all points in the lower half of the indifference curve are optimal.
- If $-a < -\frac{p_1}{p_2}$, the optimal solution will be the corner solution at $(y/p_1, 0)$.

### 0.11 1.33

Suppose we have the indirect utility function $v(\boldsymbol{p}, y)$ which is the maximized value of some utility function $u(\boldsymbol{x})$. We apply the positive, monotonic transform $f(\cdot)$ to get $f(v(\boldsymbol{p}, y))$. If we can show that this is the maximized value of some utility function $v(\boldsymbol{x})$, and that $v(\cdot)$ represents the same preferences as $u(\cdot)$ (i.e. given $(\boldsymbol{p}, y)$, the solutions for maximizing $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are the same), then we are done.

Suppose that $v(\cdot)$ is the result of applying $f(\cdot)$ to $u(\cdot)$: $v(\boldsymbol{x}) = f(u(\boldsymbol{x}))$. We know that applying a positive, monotonic transform to a utility function gives us another utility function that represents the same preferences (see Theorem 1.2 in Chapter 1). The last thing we need to show is that $f(v(\boldsymbol{p}, y))$ is the indirect utility function of $f(u(\boldsymbol{x}))$, that is, for any $(\boldsymbol{p}, y)$,

$$\max_{\boldsymbol{x}} f(u(\boldsymbol{x})) = f(v(\boldsymbol{p}, y)) = f\left(\max_{\boldsymbol{x}} u(\boldsymbol{x})\right)$$

This is true because $f(\cdot)$ is monotonic: $f(a) \geq f(b)$ iff $a \geq b$. Therefore, we have proven the statement.