Advanced Microeconomic Analysis

Solutions to Homework #2

0.1 1.41

Prove that Hicksian demands are homogeneous of degree 0 in prices. We use the relationship between Hicksian and Marshallian demands:

$$x_i^h(\boldsymbol{p}, u) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

where $e(\boldsymbol{p}, u)$ is the expenditure function. Then we use the fact that $e(\boldsymbol{p}, u)$ is homogeneous of degree 1 in $\boldsymbol{p}$, and Marshallian demand $x_i(\boldsymbol{p}, y)$ is homogeneous of degree 0 in $(\boldsymbol{p}, y)$:

$$x_i^h(t\boldsymbol{p}, u) = x_i(t\boldsymbol{p}, e(t\boldsymbol{p}, u)) = x_i(t\boldsymbol{p}, e(t\boldsymbol{p}, u)) = x_i(t\boldsymbol{p}, te(\boldsymbol{p}, u)) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

This is the same as the original value of $x_i^h(\boldsymbol{p}, u)$, so it is homogeneous of degree 0.

0.2 1.42

We use the Slutsky equation:

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_i} = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_i} - x_i(\boldsymbol{p}, u)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y}$$

The second term $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is always ≤ 0, and demand $x_i(\boldsymbol{p}, u)$ is always positive.

- Suppose x_i is a normal good. By definition, $\frac{\partial x_i}{\partial y} \geq 0$. Therefore, the sign of $-x_i(\boldsymbol{p}, u)\frac{\partial x_i(\boldsymbol{p},y)}{\partial y}$ is ≤ 0, so the sign of the left hand sign is ≤ 0. A decrease in own-price causes quantity demanded to increase.

  The converse of this statement is: if $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_j} \leq 0$, then x_i is a normal good. This depends on the relative magnitude of the two terms on the right-hand side; it may be possible for $\frac{\partial x_i}{\partial y}$ to be < 0 if the magnitude of $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is large. Therefore, the converse is not true.

- Suppose an own-price decrease causes a decrease in quantity demanded, i.e. $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_i} > 0$. Then the third term must be positive, so $\frac{\partial x_i(\boldsymbol{p},y)}{\partial y}$ must be negative, therefore x_i is an inferior good.

  The converse of this statement is: if x_i is inferior (therefore $\frac{\partial x_i(\boldsymbol{p},y)}{\partial y} < 0$), then $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_i} > 0$. This is not true if the magnitude of $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is large enough. Therefore, the converse is not true.

0.3 1.54

$$u(x_1, ..., x_n) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \sum_{i=1}^{n} \alpha_i = 1$$

$$L(x_1, ..., x_n) = Ax_1^{\alpha_1}...x_n^{\alpha_n} - \lambda(p_1x_1 + ... + p_nx_n - y)$$

$$\frac{\partial L}{\partial x_i} = \frac{\alpha_i A x_1^{\alpha_1}...x_n^{\alpha_n}}{x_i} - \lambda p_i = 0 \qquad \text{for } i = 1...n$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + ... + p_nx_n - y = 0$$

$$\frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i} = \frac{p_i}{p_j} \Rightarrow x_j = \frac{p_i}{p_j}\frac{\alpha_j}{\alpha_i}x_i$$

Plugging into the budget equation:

$$p_1x_1 + p_2\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1 + ... + p_n\frac{p_1}{p_n}\frac{\alpha_n}{\alpha_1}x_1 = y$$

Marshallian demand:

$$x_i = \frac{y}{p_i\frac{\sum_j \alpha_j}{\alpha_i}} = \frac{\alpha_i y}{p_i}$$

Indirect utility:

$$u(\boldsymbol{x}^*) = A\left(\frac{\alpha_1 y}{p_1}\right)^{\alpha}_{1} ... \left(\frac{\alpha_n y}{p_n}\right)^{\alpha}_{n} = Ay\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} ... \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

Expenditure function: use the relationship $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$

$$Ay\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} = u \Rightarrow y = \frac{u}{A}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

Hicksian demand: differentiate $e(\boldsymbol{p}, u)$ with respect to p_i.

$$x_i^h(\boldsymbol{p}, u) = \frac{\alpha_i u}{Ap_i}\prod_{j=1}^{n}\left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}$$

0.4 1.56

- $v(p_1, p_2, p_3, y) = f(y)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. In order for this to be a legitimate indirect utility, it must satisfy the following conditions (all functions must be continuous):
  - Homogeneous of degree 0 in $(\boldsymbol{p}, y)$. Then $f(y)$ must be homogeneous of degree $-(\alpha_1 + \alpha_2 + \alpha_3)$
  - Strictly increasing in y. Then $f(y)$ must be strictly increasing.
  - Decreasing in $\boldsymbol{p}$. Then $\alpha_1, \alpha_2, \alpha_3$ must be ≤ 0.
  - Quasiconvex in $(\boldsymbol{p}, y)$. Then $(\alpha_1 + \alpha_2 + \alpha_3) \geq -1$ and $f(y)$ must be convex.

- $v(p_1, p_2, y) = w(p_1, p_2) + \frac{z(p_1,p_2)}{y}$.
  - Homogeneous of degree 0 in $(\boldsymbol{p}, y)$. Then $z(p_1, p_2)$ must be homogeneous of degree 1 and $w(p_1, p_2)$ must be homogeneous of degree 0.
  - Strictly increasing in y. This is always satisfied.
  - Decreasing in $\boldsymbol{p}$. $w(p_1, p_2)$ and $z(p_1, p_2)$ must be decreasing.
  - Quasiconvex in $(\boldsymbol{p}, y)$. $w(p_1, p_2)$ and $z(p_1, p_2)$ must be quasiconvex.

0.5 2.3

Given $v(\boldsymbol{p}, y) = yp_1^\alpha p_2^\beta$, $\alpha, \beta < 0$. The direct utility is:

$$u(\boldsymbol{x}) = \min_{\boldsymbol{p}} v(\boldsymbol{p}, 1) \qquad \text{s.t. } \boldsymbol{p} \cdot \boldsymbol{x} = 1$$

$$= \min_{\boldsymbol{p}} p_1^\alpha p_2^\beta \qquad \text{s.t. } p_1x_1 + p_2x_2 = 1$$

$$L(p_1, p_2, \lambda) = p_1^\alpha p_2^\beta - \lambda(p_1x_1 + p_2x_2 - 1)$$

$$\frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta - \lambda x_1 = 0$$

$$\frac{\partial L}{\partial p_2} = \beta p_1^\alpha p_2^{\beta-1} - \lambda x_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 - 1 = 0$$

$$\frac{\alpha}{\beta} = \frac{p_1x_1}{p_2x_2} \Rightarrow p_2 = \frac{\beta}{\alpha}\frac{x_1}{x_2}, p_1 = \frac{\alpha}{\beta}\frac{x_2}{x_1}p_2$$

$$p_1x_1 + p_1\frac{x_1\beta}{x_2\alpha}x_2 = 1 \Rightarrow p_1 = \frac{1}{x_1(1+\frac{\beta}{\alpha})}, p_2 = \frac{1}{x_2(1+\frac{\alpha}{\beta})}$$

$$u(x_1, x_2) = \left(\frac{1}{x_1(1+\frac{\beta}{\alpha})}\right)^\alpha \left(\frac{1}{x_2(1+\frac{\alpha}{\beta})}\right)^\beta$$

0.6 2.16

We will show that any finite set of outcomes can be sorted into a sequence that is decreasing in preferability via induction on the number of elements. Suppose we have a sorted set of outcomes $A = \{a_1, ..., a_n\}$, such that $a_1 \succsim a_2 \succsim ... \succsim a_n$. We will show that if we add an additional element b to this set, it is possible to construct a $n+1$-length sorted set containing $a_1, ...a_n, b$. Construct the sequence of true (T) or false (F) values by comparing b to each a_i:

$$(b \succsim a_1), (b \succsim a_2), ..., (b \succsim a_n)$$

By the completeness axiom, each element is well-defined and is either T or F. By the transitivity axiom, if $b \succsim a_i$ for some i, then $b \succsim a_{i+1}, b \succsim a_{i+2}, ...b \succsim a_n$. Let $j \in \{1, ..., n\}$ be the index of the first element of A that is less preferred than b; that is, $b \succsim a_j$ is true and $b \succsim a_k$ is false for all $k < j$. We create a new sequence by inserting b at position $j-1$. Let the sequence $\{c_1, ...c_{n+1}\}$ be defined as $c_1 = a_1, c_2 = a_2, ...c_j = b, c_{j+1} = a_j, c_{j+2} = a_{j+1}, ...c_{n+1} = a_n$. This sequence is sorted, therefore it has a best and worst element.

Any sequence containing 1 element is sorted, and has a best and worst element. By the proof above, this is also true for $n = 2, 3, ...$

0.7 2.17

Suppose $a_1 \succ a_2 \succ a_3$, and the gamble $g = (1 \circ a_2)$. If $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_3)$, then α must be strictly between 0 and 1.

By the continuity axiom, we know $\alpha \in [0, 1]$ must exist. Suppose $\alpha = 0$. Then $g = (0 \circ a_1, 1 \circ a_n)\ a_2$, which is a contradiction because $a_2 \succ a_3$ by assumption. Therefore, α cannot be 0. Suppose $\alpha = 1$. Then $g = (1 \circ a_1, 0 \circ a_n)\ a_2$, which is a contradiction because $a_1 \succ a_2$ by assumption. Therefore, α cannot be 1. α must be $\in (0, 1)$.

0.8 2.25

Suppose $U(w) = a + bw + cw^2$.

- This displays risk aversion if and only if it is concave, which is true iff $c \leq 0$.
- A VNM utility function must be strictly increasing in wealth, so the region over which $U(\cdot)$ is increasing is a valid domain. This is $(-\infty, \frac{-b}{2c})$.
- Given the gamble $g = (\frac{1}{2} \circ (w+h), \frac{1}{2} \circ (w-h))$, then $E(g) = w$. We will show that $CE < E(g)$. CE satisfies the condition

  $$U(CE) = U(g) = \frac{1}{2}(a + b(w+h) + c(w+h)^2) + \frac{1}{2}(a + b(w-h) + c(w-h)^2)$$
  $$= a + bw + c(w^2 + h^2)$$

  Since $u(\cdot)$ is strictly increasing, if $u(x) > u(y)$, then $x > y$. $U(E(g)) = U(w) = a + bw + c(w^2)$, which is strictly less than $U(CE) = a + bw + c(w^2 + h^2)$ if $h > 0$. Therefore, $CE < E(g)$. Since $P = E(g) - CE$, then $P > 0$.
- For this utility function, $R_a(w) = \frac{-u''(w)}{u'(w)} = \frac{-2c}{b+2cw}$, which is increasing in w. Therefore, this utility cannot represent preferences with decreasing absolute risk aversion.

0.9 2.32

Suppose a VNM utility function displays constant absolute risk aversion, so that $R_a(w) = \frac{-u''(w)}{u'(w)} = \alpha$ for all w. Then $-\alpha u'(w) = u''(w)$ for all w, or $\frac{d}{dw}u'(w) = -\alpha u'(w)$ for all w. The only functional form for $u'(w)$ that satisfies this is the exponential form, $u(x) = e^{-\alpha w}$. Then $u'(w) = -\alpha e^{-\alpha w}, u''(w) = \alpha^2 e^{-\alpha w}$, and $R_a(w) = \alpha$.

0.10 Q10

Suppose $u(w) = \ln(w)$.

(a) Find the Arrow-Pratt measure of absolute risk aversion. Is the utility function CARA, DARA, or IARA?

$$R_a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-1/w^2}{1/w} = \frac{1}{w}$$

$R_a(w)$ is decreasing in w, so this is DARA.

Suppose a consumer has an initial wealth of w_0 and is choosing a fraction x of his wealth, where $0 \leq x \leq 1$, to invest in a risky asset. The risky asset has two outcomes: with probability p, it will give a return of 0 (a total loss), and with probability $1-p$, it will give a return of r, so that if amount xw_0 is invested, the total return is rxw_0. The portion of wealth not invested in the risky asset is stored as cash, which has a certain return of 100%. Assume that the expected return is positive.

(b) In each of the two possible outcomes, what is the wealth of the consumer?

In the "bad" outcome, total wealth is $(1-x)w_0$. In the "good" outcome, total wealth is $(1-x)w_0 + rxw_0$.

(c) Write down the expected wealth of the consumer, as a function of x.

Expected wealth is $p((1-x)w_0) + (1-p)((1-x)w_0 + rxw_0) = w_0(1 + x(r(1-p) - 1))$.

(d) Write down the expected utility of the consumer, as a function of x.

Expected utility is

$$E[u(w)] = p\ln((1-x)w_0) + (1-p)\ln((1-x)w_0 + rxw_0)$$

(e) Find the value of x that maximizes the expected utility of the consumer.

Taking the derivative with respect to x and setting it to 0, we get:

$$\frac{\partial E[u(w)]}{\partial x} = \frac{p}{x-1} + \frac{(1-p)(w_0(r-1))}{w_0(1-x+rx)} = 0$$

Solving for x, we get $x = \frac{r-1-pr}{r-1}$. This is increasing in $1-p$ and r.