Advanced Microeconomic Analysis

Solutions to Homework #3

0.1 4.20

The compensating variation (CV) is defined as the amount of income that must be given to the consumer to keep his utility the same after a price change. Since there is only one good, the consumer's utility is completely determined by the quantity of the single good, given by $x(p, y) = y/p$. Therefore, to keep utility the same, the quantity of the good must be the same. Before the price change, the quantity consumed is $y/p = 7/1 = 7$. After the price change, the consumer requires a total income of $4 \cdot 7 = 28$ to buy a quantity of 7, which means the CV is $28 - 7 = 21$.

0.2 4.22

(a) Monopolist's marginal cost is c. Revenue is $pq = (\alpha - \beta q)q = \alpha q - \beta q^2$. Marginal revenue is $\alpha - 2\beta q$. The optimality condition for a monopolist is $MR = MC$, or $c = \alpha - 2\beta q \Rightarrow q^* = \frac{\alpha - c}{2\beta}$. Market price is $\alpha - \frac{\alpha - c}{2} = (\alpha - c)/2$. Profits are $pq - (cq + F) = \frac{(\alpha - c)^2}{4\beta} - F$.

(b) The deadweight loss is the loss in total surplus compared to the perfect competition outcome, where output is chosen such that $P = MC$. Let's compute consumer and producer surplus in both cases. Producer surplus is equal to profits plus fixed cost.

(i): perfect competition. $p = c, q = (\alpha - c)/\beta$. Consumer surplus $= \frac{1}{2}(\alpha - c)((\alpha - c)/\beta)$. Producer surplus is zero, since marginal cost is constant. Total surplus is $\frac{1}{2}(\alpha - c)((\alpha - c)/\beta)$.

(ii): monopoly. $p = (\alpha - c)/2, q = \frac{\alpha - c}{2\beta}$. Consumer surplus $= \frac{1}{2}(\alpha - (\alpha - c)/2))(\frac{\alpha - c}{2\beta}) = \frac{\alpha^2 - c^2}{8\beta}$. Producer surplus $= \frac{(\alpha - c)^2}{4\beta}$. Total surplus is $\frac{(\alpha - c)(\alpha - 3c)}{8\beta}$.

Deadweight loss is the difference in total surplus, which is $\frac{(\alpha - c)(\alpha - 3c)}{8\beta}$.

(c) Total surplus is maximized at the perfectly competitive outcome, which is when $p = c$. The firm's profits will be $-F$.

0.3 5.4

In example 5.1, there are two consumers with CES utility functions $u^i(x_1, x_2) = x_1^\rho + x_2^\rho$. Endowments are: $\boldsymbol{e}^1 = (1, 0)$ and $\boldsymbol{e}^2 = (0, 1)$. Consumer i's demand for good j will be

$$x_j^i(p_1, p_2, y^i) = \frac{p_j^{r-1} y^i}{p_1^r + p_2^r}$$

where $r = \rho/(\rho - 1)$. $y^1 = \boldsymbol{p} \cdot \boldsymbol{e}^1 = p_1$ and $y^2 = \boldsymbol{p} \cdot \boldsymbol{e}^2 = p_2$, therefore

$$x_j^1(p_1, p_2) = \frac{p_j^{r-1} p_1}{p_1^r + p_2^r}, \qquad x_j^2(p_1, p_2) = \frac{p_j^{r-1} p_2}{p_1^r + p_2^r}$$

Aggregate excess demand is given by

$$z_j(p_1,p_2) = x_j^1(p_1,p_2,y^i) + x_j^2(p_1,p_2,y^i) - 1 = \frac{p_j^{r-1}p_1}{p_1^r + p_2^r} + \frac{p_j^{r-1}p_2}{p_1^r + p_2^r} - 1$$
$$= \frac{p_j^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1$$

Walras' law is:

$$p_1z_1 + p_2z_2 = 0$$
$$= p_1\left(\frac{p_1^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1\right) + p_2\left(\frac{p_2^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1\right)$$
$$= \frac{p_1^r(p_1+p_2)}{p_1^r + p_2^r} + \frac{p_2^r(p_1+p_2)}{p_1^r + p_2^r} - p1 - p2$$
$$= \frac{(p_1^r + p_2^r)(p_1+p_2)}{p_1^r + p_2^r} - p_1 - p_2 = 0$$

0.4 5.5

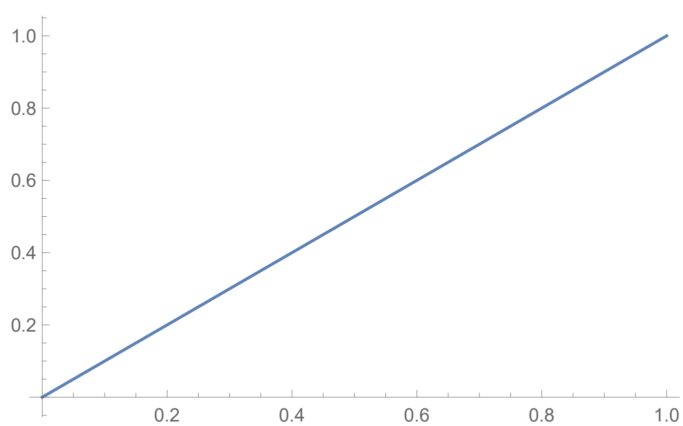
Each consumer's MRS is:

$$MRS_i = \frac{\rho x_1^{\rho-1}}{\rho x_2^{\rho-1}} = \left(\frac{x_1}{x_2}\right)^{\rho-1}$$

The contract curve is defined by:

$$\left(\frac{x_1}{x_2}\right)^{\rho-1} = \left(\frac{1-x_1}{1-x_2}\right)^{\rho-1}$$
$$\Rightarrow \frac{x_1}{x_2} = \frac{1-x_1}{1-x_2} \qquad \Rightarrow \qquad \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

Since $x/(1-x)$ is a strictly increasing function, equality implies that $x_1 = x_2$. Therefore, the contract curve is the 45-degree line.



At every point on the contract curve, $MRS_1 = MRS_2 = 1$. Therefore, the Walrasian equilibrium, given an initial allocation of $\boldsymbol{e}^1 = (1, 0)$, must have a price line with slope 1 that goes through $(1, 0)$. The intersection of this line with the contract curve is at $(1/2, 1/2)$, which is the Walrasian equilibrium allocation.

0.5 5.11

(a) First, let's characterize the set of Pareto-efficient allocations. This is the contract curve, i.e. the set of allocations where the indifference curves are tangent, or where $MRS^1 = MRS^2$.

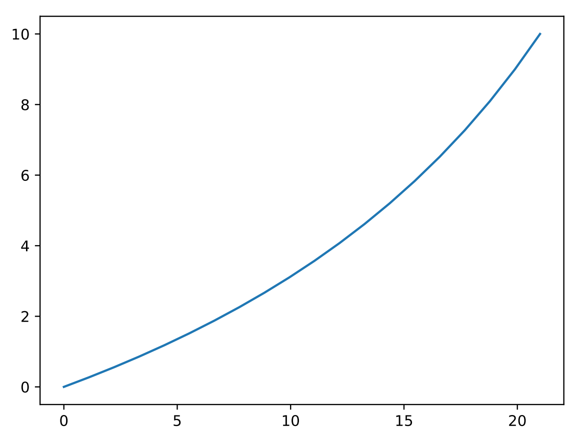
$$MRS^1 = \frac{2x_1^1(x_2^1)^2}{2(x_1^1)^2x_2^1} = \frac{x_2^1}{x_1^1}, \qquad MRS^2 = \frac{1/x_1^2}{2/x_2^2} = \frac{1}{2}\frac{x_2^2}{x_1^2}$$

$$\frac{x_2^1}{x_1^1} = \left(\frac{1}{2}\right)\frac{10 - x_2^1}{21 - x_1^1}$$

$$\frac{10 - x_2^1}{x_2^1} = 2\frac{21 - x_1^1}{x_1^1}$$

$$\frac{10}{x_2^1} - 1 = \frac{42}{x_1^1} - 2$$

$$x_2^1 = \frac{10x_1^1}{42 - x_1^1}$$

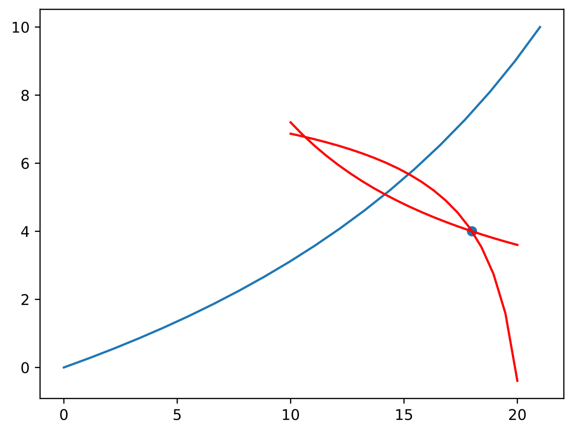


(b) Now, let's find the core, which is the part of the contract curve that gives a higher utility for both consumers. Plugging in the endowments of each consumer, we get:

$$u^1(18, 4) = 5184, \qquad u^2(3, 6) = 4.68213$$

So we can say that the core is the part of the curve $x_2^1 = \frac{10x_1^1}{42 - x_1^1}$, such that

$$(x_1^1x_2^1)^2 \geq 5184 \qquad \text{and} \qquad \ln(21 - x_1^1) + 2\ln(10 - x_2^1) \geq 4.68213$$



(c) Now, we want to find a Walrasian equilibrium. We want to find prices p_1, p_2 such that aggregate excess demand is zero. Since only relative prices matter, let's set $p_2 = 1$ and solve for p_1. We will need to find the Marshallian demand function for each consumer. To speed things up, we'll use a shortcut. Recall from Chapter 1 that an increasing transformation of a utility function represents the same preferences. Therefore, if we have two utility functions, $u_1(x), u_2(x)$ where $u_1(x) = f(u_2(x))$ and $f(\cdot)$ is an increasing function, then these represent the same preferences. Marshallian demand for and the expenditure function will be unchanged (but obviously, indirect utility will be different).

Note that:

- $u^1(x_1, x_2) = (x_1 x_2)^2 = (x_1^{1/2} x_2^{1/2})^4$
- $u^2(x_1, x_2) = \ln(x_1) + 2\ln(x_2) = 3\ln\left(x_1^{1/3} x_2^{2/3}\right)$

That is, both u^1 and u^2 are transformations of Cobb-Douglas utility functions (as you might have guessed from the MRS). Marshallian demand for the Cobb-Douglas utility $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ is:

$$x_1^*(p_1, p_2, y) = \frac{\alpha y}{p_1}, \qquad , x_2^*(p_1, p_2, y) = \frac{(1-\alpha)y}{p_2}$$

Plugging in $p_2 = 1$, $y^1 = \boldsymbol{p} \cdot \boldsymbol{e}^1 = p_1 18 + 4$, $y^2 = \boldsymbol{p} \cdot \boldsymbol{e}^2 = p_1 3 + 6$, we get

$$x_1^1 = \frac{(1/2)(p_1 18 + 4)}{p_1}, \qquad x_2^1 = \frac{(1/2)(p_1 18 + 4)}{1}$$
$$x_1^2 = \frac{(1/3)(p_1 3 + 6)}{p_1}, \qquad x_2^2 = \frac{(2/3)(p_1 3 + 6)}{1}$$

We want to find p_1 such that $x_1^1 + x_1^2 = 18 + 3 = 21$, and $x_2^1 + x_2^2 = 6 + 4 = 10$. These conditions (total demand = total supply) are called *market clearing* conditions. We can solve each equation separately: it should give the same result. Let's try the market clearing condition for good x_2.

$$
\begin{aligned}
x_2^1 + x_2^2 &= (1/2)(p_1 18 + 4) + (2/3)(p_1 3 + 6) = 10 \\
\frac{18p_1 + 4}{2} + \frac{2(3p_1 + 6)}{3} &= 10 \\
54p_1 + 12 + 12p_1 + 24 &= 60 \\
66p_1 = 24 \qquad \Rightarrow \qquad p_1 &= \frac{4}{11}
\end{aligned}
$$

So, any p_1, p_2 such that $p_1/p_2 = 4/11$ is a Walrasian price that sets aggregate excess demand to zero. The Walrasian equilibrium allocation that results from $p_1 = 4/11, p_2 = 1$ is:

$$x_1^1 = 29/2,\ x_2^1 = 58/11,\ x_1^2 = 13/2,\ x_2^2 = 52/11$$

(d) Since each agent is solving his utility maximization problem, his MRS is equal to the price ratio, and therefore equal to the other agent's MRS. Each agent is maximizing utility, and by definition, cannot end up at a lower utility than what he started with (at his endowment). Therefore, the WEA must be in the core.

0.6 5.18

(a) Note that $x_1^1 = 20 - x_1^2$ and $x_2^1 = 10 - x_2^2$. We will use x_1^2, x_2^2 as our choice variables, and cast the problem as a contstrained optimization problem:

$$\max_{x_1, x_2} (20 - x_1)(10 - x_2)^2 \qquad \text{s.t. } (x_1)^2(x_2) = \frac{8000}{27}$$

We form the Lagrangian:

$$L(x_1, x_2, \lambda) = (20 - x_1)(10 - x_2)^2 - \lambda((x_1)^2(x_2) - \frac{8000}{27})$$

First order conditions are:

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= -(10 - x_2)^2 - 2\lambda(x_1)(x_2) = 0 \\
\frac{\partial L}{\partial x_2} &= -2(20 - x_1)(10 - x_2) - \lambda(x_1)^2 0 \\
\frac{\partial L}{\partial \lambda} &= ((x_1)^2(x_2) - \frac{8000}{27}) = 0
\end{aligned}
$$

Rearranging the first two equations and dividing the first one by the second one, we get

$$\frac{1}{2}\frac{10 - x_2}{20 - x_1} = 2\frac{x_2}{x_1}$$

Note that this is simply specifying that the MRS of $u^1(\cdot)$ is equal to the MRS of $u^2(\cdot)$, and in fact u^1 and u^2 are transformations of Cobb-Douglas utility. Therefore, we know that any allocation that satisfies the first-order conditions is on the contract curve, and therefore Pareto-optimal. Rearrange $(x_1)^2(x_2) = \frac{8000}{27}$ to get $x_2 = \frac{8000}{27x_1^2}$, and plug into:

$$\begin{aligned}
4\frac{x_2}{x_1} &= \frac{20-x_2}{10-x_1} \\
\frac{4\frac{8000}{27x_1^2}}{x_1} &= \frac{20-\frac{8000}{27x_1^2}}{10-x_1} \\
\frac{32000}{27x_1} &= \frac{540x_1^2-8000}{27(10-x_1)} \\
540x_1^3 &+ 24000x_1 - 320000 = 0
\end{aligned}$$

The only real solution to this cubic equation is $x_1 = 20/3$, giving $x_1^1 = 10/3, x_2^1 = 40/3, x_1^2 = 20/3, x_2^2 = 20/3$. We can verify that $u^2(20/3, 20/3) = 8000/27$.

(b) Suppose that the Walrasian equilibrium given an initial allocation of $\boldsymbol{e}^1 = (10,0), \boldsymbol{e}^2 = (0,20)$ is the solution in the previous question, $x_1^1 = 40/3, x_2^1 = 10/3$. The price line must go through both points, and has a slope of $(40/3-0)/(20-10/3) = 2$. We can verify that this is equal to the MRS of both agents at the solution:

$$MRS_1 = \frac{1}{2}\frac{40/3}{10/3} = 2, \qquad MRS_2 = 2\frac{20/3}{20/3} = 2$$

Therefore, the price line is tangent to the indifference curves of both agents, and they are solving their utility maximization problem at the allocation. Therefore, the allocation is a Walrasian equilibrium.

0.7 Q7

Let x_1, x_2 denote the amount of wealth in state $s = 1, s = 2$ respectively. Expected utility is $E(u) = p\ln(x_1) + (1-p)\ln(x_2)$. Note that this is an increasing transformation of a Cobb-Douglas utility:

$$p\ln(x_1) + (1-p)\ln(x_2) = \ln(x_1^p x_2^{1-p})$$

Therefore, demand will be the same as Cobb-Douglas:

$$x_1^* = \frac{py}{p_1}, \qquad x_2^* = \frac{(1-p)y}{p_2}$$

Since only relative prices matter, fix $p_2 = 1$. Income of each agent at market prices p_1, p_2 is then:

$$y^1 = p_1(1) + p_2(3) = p_1 + 3, \qquad y^2 = p_1(3) + p_2(1) = 3p_1 + 1$$

The market clearing conditions are: $x_1^1 + x_1^2 = 4, x_2^1 + x_2^2 = 4$, or

$$\frac{p(p_1+3)}{p_1} + \frac{p(3p_1+1)}{p_1} = 4, \qquad \frac{(1-p)(p_1+3)}{1} + \frac{(1-p)(3p_1+1)}{1} = 4$$

Either of these equations can be used to solve for p_1. Using the second one, we get $p_1 = \frac{p}{1-p}$. Therefore, any prices p_1, p_2 such that $p_1/p_2 = p/(1-p)$ is a Walrasian equilibrium.

(a) Suppose $p = 1/2$. Then $p_1 = 1, p_2 = 1, y^1 = 4, y^2 = 4, x_1^1 = 2, x_2^1 = 2, x_1^2 = 2, x_2^2 = 2$.

(b) Suppose $p = 1/3$. Then $p_1 = 1/2, p_2 = 1, y^1 = 7/2, y^2 = 5/2, x_1^1 = 7/3, x_2^1 = 7/3, x_1^2 = 5/3, x_2^2 = 5/3$.

Note that both agents have eliminated all risk, i.e. their wealth is the same in each state.