Advanced Microeconomic Analysis

Midterm Exam

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Q1. **(40 pts)** Suppose a consumer has the utility function

$$u(x_1, x_2, x_3) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3)$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$. Prices are p_1, p_2, p_3 and consumer wealth is w.

(a) **(10 pts)** Find the Marshallian demand functions and indirect utility function.

The Lagrangian is:

$$L(p_1, p_2, p_3, \lambda) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - w)$$

First order conditions are:

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_1}{x_1} - \lambda p_1 = 0 \to p_1x_1 = \frac{\alpha 1}{\lambda}$$

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_2}{x_2} - \lambda p_2 = 0 \to p_2x_2 = \frac{\alpha 2}{\lambda}$$

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_3}{x_3} - \lambda p_3 = 0 \to p_3x_3 = \frac{\alpha 3}{\lambda}$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 + p_3x_3 - w = 0$$

Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. Solving for x_1, x_2, x_3, λ, we get:

$$p_1x_1 + p_2x_2 + p_3x_3 = w = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\lambda} \to \lambda = \frac{\alpha}{w}$$

$$x_1 = \frac{\alpha_1 w}{p_1 \alpha}, x_2 = \frac{\alpha_2 w}{p_2 \alpha}, x_1 = \frac{\alpha_3 w}{p_3 \alpha}$$

The indirect utility function is

$$v(p_1, p_2, p_3, w) = \alpha_1 \ln(\frac{\alpha_1 w}{p_1 \alpha}) + \alpha_2 \ln(\frac{\alpha_2 w}{p_2 \alpha}) + \alpha_3 \ln(\frac{\alpha_3 w}{p_3 \alpha})$$

$$= \alpha \ln(w) + \alpha_1 \ln(\frac{\alpha_1}{p_1 \alpha}) + \alpha_2 \ln(\frac{\alpha_2}{p_2 \alpha}) + \alpha_3 \ln(\frac{\alpha_3}{p_3 \alpha})$$

(b) **(10 pts)** Find the expenditure function and Hicksian demand functions.

Using the equation $u = v(p, e(p, u)))$, we get:

$$e(p_1, p_2, p_3, u) = \exp\left(\frac{u - (\alpha_1 \ln(\frac{\alpha_1}{p_1 \alpha}) + \alpha_2 \ln(\frac{\alpha_2}{p_2 \alpha}) + \alpha_3 \ln(\frac{\alpha_3}{p_3 \alpha}))}{\alpha}\right)$$

The Hicksian demand functions are given by $\frac{\partial e}{\partial p_i}$:

$$x_i^h(p_1, p_2, p_3, u) = \frac{\alpha_i e(p_1, p_2, p_3, u)}{\alpha p_i}$$

Now, for the following questions, suppose there is a additional *rationing* constraint: the consumer cannot buy more than k units of good 1, where $k > 0$.

(c) **(5 pts)** Write down the Lagrangian and the Kuhn-Tucker conditions (that is, necessary conditions for optimality) for this problem.

The optimization problem is

$$\max_{x_1,x_2,x_3} u(x_1, x_2, x_3) \qquad \text{s.t. } p_1x_1 + p_2x_2 + p_3x_3 - w \leq 0, x_1 - k \leq 0$$

We introduce an additional Lagrange multiplier, μ, for the constraint $x_1 - k \leq 0$. The Lagrangian is:

$$L(p_1, p_2, p_3, \lambda, \mu) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - w) - \mu(x_1 - k)$$

The Kuhn-Tucker constraints are simply the first-order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_1}{x_1} - \lambda p_1 - \mu = 0$$

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_2}{x_2} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_3}{x_3} - \lambda p_3 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 + p_3x_3 - w = 0$$

$$\frac{\partial L}{\partial \mu} = x_1 - k = 0$$

combined with the "complementary slackness" conditions

$$\lambda \geq 0, \lambda(p_1x_1 + p_2x_2 + p_3x_3 - w) = 0$$

$$\mu \geq 0, \mu(x_1 - k) = 0$$

which state that the Lagrange multiplier is positive if and only if the associated constraint is binding (i.e. satisfied with strict equality).

(d) **(5 pts)** Under what conditions on the parameters is the rationing constraint binding?

The rationing constraint will be binding when the consumer wants to buy $x_1 \geq k$, which is when

$$\frac{\alpha_1 w}{p_1 \alpha} \geq k$$

(e) **(5 pts)** Suppose the indirect utility function is $v(p_1, p_2, p_3, w, k)$. Find the shadow prices of w and k (i.e. the derivative of v with respect to w and k) as a function of the parameters when

- the rationing constraint is not binding
- the rationing constraint is binding.

As shown in Exercise A2.33, the shadow prices are equal to the Lagrange multipliers on the corresponding constraint:

$$\frac{\partial v}{\partial w} = \lambda, \frac{\partial v}{\partial k} = \mu$$

A constraint that is non-binding has a multiplier of 0. When the rationing constraint is not binding ($x_1 < k$), λ is the same as before and $\mu = 0$. When the rationing constraint is binding, $x_1 = k$ and:

$$\lambda = \frac{\alpha_2 + \alpha_3}{w - kp_1}, \mu = \frac{\alpha_1}{k} - \lambda p_1$$

(f) **(5 pts)** Suppose $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}, p_1 = p_2 = 1, p_3 = 2$, $k = 1$, and $w = 10$. Find the optimal x_1, x_2, x_3.

The rationing constraint will be binding, so $x_1 = 1$. 9 wealth is left over for x_2 and x_3. The first-order constraints imply that $p_2x_2 = p_3x_3$, giving $x_2 = 6, x_3 = 3$.

Q2. **(25 pts)** A firm has production function $f(x_1, x_2) = (x_1 - a)^{\frac{1}{4}}(x_2 - b)^{\frac{3}{4}}$, where $a, b \geq 0$. Suppose the prices of inputs x_1, x_2 are w_1, w_2 respectively.

(a) **(10 pts)** Find the cost function when $a = b = 0$.

(b) **(10 pts)** Find the cost function for general a and b.

(c) **(5 pts)** Find the conditional input demand function for x_1, for general a and b.

Let the problem be

$$\min_{x_1,x_2} w_1x_1 + w_2x_2 \qquad \text{s.t. } (x_1 - a)^{\alpha_1}(x_2 - b)^{\alpha_2} = y$$

where $\alpha_1 + \alpha_2 = 1$. The first-order conditions are:

$$w_1 = \frac{\lambda\alpha_1 y}{x_1 - a} \to (x_1 - a) = \frac{\lambda\alpha_1 y}{w_1}$$

$$w_2 = \frac{\lambda\alpha_2 y}{x_2 - b} \to (x_2 - b) = \frac{\lambda\alpha_2 y}{w_2}$$

Multiplying $(x_1 - a)^{\alpha_1}$ and $(x_2 - b)^{\alpha_2}$, we get

$$(x_1 - a)^{\alpha_1}(x_2 - b)^{\alpha_2} = y = \left(\frac{\lambda\alpha_1 y}{w_1}\right)^{\alpha_1}\left(\frac{\lambda\alpha_2 y}{w_2}\right)^{\alpha_2}$$

$$\lambda = \frac{w_1^{\alpha_1} w_2^{\alpha_2}}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}$$

The conditional input demand functions are given by:

$$x_1 = \frac{w_1^{\alpha_1 - 1} w_2^{\alpha_2} y}{\alpha_1^{\alpha_1 - 1} \alpha_2^{\alpha_2}} + a$$

$$x_2 = \frac{w_1^{\alpha_1} w_2^{\alpha_2 - 1} y}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2 - 1}} + b$$

The cost function is

$$w_1 x_1 + w_2 x_2 = w_1 \left(\frac{w_1^{\alpha_1 - 1} w_2^{\alpha_2} y}{\alpha_1^{\alpha_1 - 1} \alpha_2^{\alpha_2}} + a \right) + w_2 \left(\frac{w_1^{\alpha_1} w_2^{\alpha_2 - 1} y}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2 - 1}} + b \right)$$

For $a = b = 0$, this is

$$w_1 x_1 + w_2 x_2 = \frac{w_1^{\alpha_1} w_2^{\alpha_2} y}{\alpha_1^{\alpha_1 - 1} \alpha_2^{\alpha_2}} + \frac{w_1^{\alpha_1} w_2^{\alpha_2} y}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2 - 1}}$$

Q3. **(20 pts)** Consider the game given by the following matrix. x is a real number.

	L	C	R
T	4,4	6,2	0,0
M	2,6	8,8	0,0
B	0,0	0,0	$x, -x$

(a) **(2 pts)** For what values of $x \in \mathbb{R}$ is B a strictly dominated strategy for Player 1?

B is strictly dominated when $x < 0$.

(b) **(8 pts)** For all possible values of $x \in \mathbb{R}$, find the outcomes that survive iterative elimination of strictly dominated strategies.

If $x < 0$ or $x > 0$, then B and R will be eliminated by iterative elimination of strictly dominated strategies, and $\{T, M\} \times \{L, C\}$ remains.

If $x = 0$, then there are no strictly dominated strategies, and the entire game remains.

(c) **(10 pts)** For all possible values of $x \in \mathbb{R}$, find the set of pure strategy Nash equilibria.

Since a strictly dominated strategy will not be played in NE, then if $x < 0$ or $x > 0$, the game is reduced to $\{T, M\} \times \{L, C\}$, which has two pure NE: (T, L) and (M, C).

If $x = 0$, then the pure NE are (T, L), (M, C), and (B, R).

	L	R
T	4,4	0,2
M	2,0	2,2
B	3,0	1,0

Q4. **(15 pts)** Consider the following game:

(a) **(5 pts)** Find the set of pure strategy Nash equilibria.

(T, L) and (M, R)

(b) **(10 pts)** Find a mixed strategy Nash equilibrium in which Player 1 plays all three actions with positive probability.

Let q be Player 2's probability of playing L. The expected payoffs to each of Player 1's actions are:

$$E_1(T) = 4q$$

$$E_1(M) = 2$$

$$E_1(B) = 3q + 1(1 - q) = 2q + 1$$

In a mixed strategy NE where all actions are played with positive probability, these expected values must be equal to each other. Solving $4q = 2 = 2q + 1$ gives $q = \frac{1}{2}$. Likewise, let $p_1, p_2, 1 - p_1 - p_2$ be the probabilities that Player 1 places on T, M, B respectively. We know that Player 2 places positive probability on both of his actions, so expected payoffs must be equal:

$$E_2(L) = 4p_1$$

$$E_2(R) = 2p_1 + 2p_2$$

$4p_1 = 2p_1 + 2p_2$ gives us $p_1 = p_2$. Therefore, a mixed strategy profile is a MSNE if it satisfies this form with $0 < p < \frac{1}{2}$:

$$\left((p, p, 1 - 2p), (\frac{1}{2}, \frac{1}{2})\right)$$