Advanced Microeconomic Analysis

Midterm Exam

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Instructions:

- Please write your name in English.
- This exam is open-book.
- Total time: 120 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(20 pts)** Suppose a consumer has the utility function

$$u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$$

where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 \leq 1$. Prices are p_1, p_2 and consumer wealth is y.

(a) **(10 pts)** Find the Marshallian demand functions and indirect utility function.

The Lagrangian is:

$$L(p_1, p_2, \lambda) = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda(p_1 x_1 + p_2 x_2 - w)$$

First order conditions are:

$$\frac{\partial L}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda p_1 = 0 \to \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} = \lambda p_1$$
$$\frac{\partial L}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} - \lambda p_2 = 0 \to \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 2} = \lambda p_2$$
$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - y = 0$$

Dividing the first equation by the second, we get the optimality condition MRS = price ratio:

$$\frac{\alpha_1 x_2}{\alpha_2 x_1} = \frac{p_1}{p_2} \qquad \to \qquad p_2 x_2 = \frac{\alpha_2}{\alpha_1} p_1 x_1$$

Plugging into the budget equation, we get

$$p_1 x_1 + \frac{\alpha_2}{\alpha_1} p_1 x_1 = w \qquad \to \qquad p_1 x_1 (1 + \frac{\alpha_2}{\alpha_1}) = y$$

$$x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{y}{p_1}$$

By symmetry, the solution for x_2 is

$$x_2^* = \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{y}{p_2}$$

The indirect utility function is:

$$(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{y}{p_1})^{\alpha_1}(\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{y}{p_2})^{\alpha_2}$$
$$= y^{\alpha_1+\alpha_2}(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{1}{p_1})^{\alpha_1}(\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{1}{p_2})^{\alpha_2}$$

(b) **(10 pts)** Find the expenditure function and Hicksian demand functions.

Using the equation $u = v(p, e(p, u)))$, we get:

$$u = e(p,u)^{\alpha_1+\alpha_2}(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{1}{p_1})^{\alpha_1}(\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{1}{p_2})^{\alpha_2}$$

$$e(p,u) = \left(u(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{1}{p_1})^{-\alpha_1}(\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{1}{p_2})^{-\alpha_2}\right)^{\frac{1}{\alpha_1+\alpha_2}}$$
$$= u^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}$$

The Hicksian demand functions are given by $\frac{\partial e}{\partial p_i}$:

$$x_1^h(p_1,p_2,u) = u^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)^{\frac{-\alpha_2}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} p_1^{\frac{-\alpha_2}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}$$

$$x_2^h(p_1,p_2,u) = u^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\frac{-\alpha_1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{-\alpha_1}{\alpha_1+\alpha_2}}$$

Q2. **(26 pts)** Suppose a consumer lives for two periods, $t = 1, 2$, and in each period, has a utility function over consumption $u(c) = \frac{c^\alpha}{\alpha}$, where $\alpha < 1, \alpha \neq 0$. Let c_1, c_2 denote the consumer's consumption in $t = 1, 2$ respectively; then the overall utility of the consumer is

$$U(c_1, c_2) = u(c_1) + u(c_2)$$

Suppose that the consumer has an initial wealth of w_0, and must choose a fraction $x, 0 \leq x \leq 1$ of his initial wealth to invest in a risky asset. If the amount invested in the risky asset is xw_0, there are two possible outcomes: (i) with probability p, the asset will return xw_0r_1; (ii) with probability $(1-p)$, the asset will return xw_0r_2, where $0 < r_1 < r_2$. The consumer will consume the amount not invested, and will consume the net return of the risky asset at $t = 2$. Therefore,

$$c_1 = (1-x)w_0, \qquad \tilde{c}_2 = xw_0\tilde{r}$$

where $\tilde{c}_2, \tilde{r}$ indicates that c_2 and the return is a random variable with two possible outcomes.

(a) **(5 pts)** Find the Arrow-Pratt measure of absolute risk aversion for $u(c)$. Is it increasing, constant, or decreasing?

$$R_a(c) = -\frac{u''(c)}{u'(c)} = -\frac{(\alpha-1)c^{\alpha-2}}{c^{\alpha-1}} = \frac{1-\alpha}{c}$$

ARA is increasing in c.

(b) **(3 pts)** For each of the two possible outcomes, write down the value of c_2.

(i) xw_0r_1; (ii) xw_0r_2

(c) **(3 pts)** Find the expected value of c_2.

$$E[c_2] = pxw_0r_1 + (1-p)xw_0r_2 = xw_0(p(r_1) + (1-p)r_2)$$

(d) **(5 pts)** Write down the expected utility, $E[U(c_1, c_2)]$ as a function of x.

$$E[U(c_1,c_2)] = u((1-x)w_0) + pu(xw_0r_1)(1-p)u(xw_0r_2)$$
$$= \frac{((1-x)w_0)^\alpha}{\alpha} + p\frac{(xw_0r_1)^\alpha}{\alpha} + (1-p)\frac{(xw_0r_2)^\alpha}{\alpha}$$

(e) **(10 pts)** Find the value of x that maximizes expected utility.

Setting the derivative with respect to x to 0 gives:

$$\frac{\partial}{\partial x} = -w_0((1-x)w^0)^{\alpha-1} + pw_0r_1(xw_0r_1)^{\alpha-1} + (1-p)w_0r_2(xw_0r_2)^{\alpha-1} = 0$$

$$(1-x)^{\alpha-1} = x^{\alpha-1}(pr_1^\alpha + (1-p)r_2^\alpha$$

$$\left(\frac{1-x}{x}\right)^{\alpha-1} = pr_1^\alpha + (1-p)r_2^\alpha = E(\tilde{r}^\alpha)$$

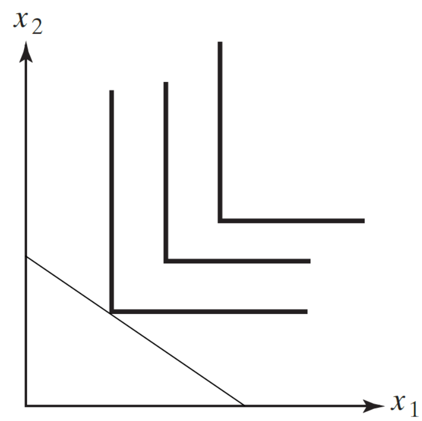
$$\frac{1-x}{x} = E(\tilde{r}^\alpha)^{\frac{1}{\alpha-1}}$$

$$1-x = xE(\tilde{r}^\alpha)^{\frac{1}{\alpha-1}}$$

$$x = (1+E(\tilde{r}^\alpha)^{\frac{1}{\alpha-1}})^{-1}$$

Q3. **(24 pts)** For each of the following utility functions, find the indirect utility function $v(p, y)$ and expenditure function $e(p, u)$:

(a) **(8 pts)** $u(x_1, x_2) = \min(x_1, x_2)$



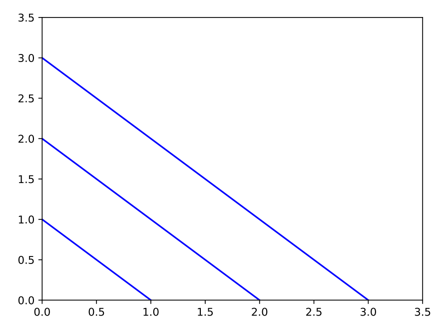
These are Leontieff preferences. For any p_1, p_2, the solution will satisfy $x_1^* = x_2^*$. Plugging into the budget equation $p_1x_1 + p_2x_2 = y$, we get

$$p_1x_1 + p_2x_1 = y \qquad \rightarrow \qquad x_1^* = \frac{y}{p_1 + p_2},\ x_2^* = \frac{y}{p_1 + p_2}$$

Therefore, $v(p_1, p_2, y) = \min(x_1^*, x_2^*) = \frac{y}{p_1+p_2}$. Applying the equation $u = v(p, e(p, u)))$, we get

$$u = \frac{e(p_1, p_2, u)}{p_1 + p_2} \qquad \rightarrow \qquad e(p_1, p_2, u) = u(p_1 + p_2)$$

(b) **(8 pts)** $u(x_1, x_2) = x_1 + x_2$



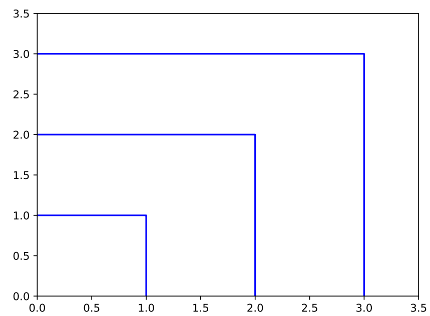
The MRS (slope of the indifference curve) is 1. If $p_1/p_2 < 1$, the solution will be at the lower right corner, where $x_1^* = y/p_1, x_2^* = 0$. If $p_1/p_2 = 1$, any point on the budget line is a solution. If $p_1/p_2 > 1$, the solution will be at the upper left corner, where $x_1^* = 0, x_2^* = y/p_2$. Therefore, the indirect utility is:

$$v(p_1, p_2, y) = \begin{cases} y/p_1 & \text{if } p_1/p_2 \leq 1 \\ y/p_2 & \text{if } p_1/p_2 > 1 \end{cases}$$

The expenditure function is

$$e(p_1, p_2, u) = \begin{cases} up_1 & \text{if } p_1/p_2 \leq 1 \\ up_2 & \text{if } p_1/p_2 > 1 \end{cases}$$

(c) **(8 pts)** $u(x_1, x_2) = \max(x_1, x_2)$



The cases are the same as in the previous case: If $p_1/p_2 < 1$, the solution will be at the lower right corner, where $x_1^* = y/p_1, x_2^* = 0$. If $p_1/p_2 = 1$, both the lower right and upper left corners are solutions. If $p_1/p_2 > 1$, the solution will be at the upper left corner, where $x_1^* = 0, x_2^* = y/p_2$. Therefore, the indirect utility is:

$$v(p_1, p_2, y) = \begin{cases} y/p_1 & \text{if } p_1/p_2 \leq 1 \\ y/p_2 & \text{if } p_1/p_2 > 1 \end{cases}$$

The expenditure function is

$$e(p_1, p_2, u) = \begin{cases} up_1 & \text{if } p_1/p_2 \leq 1 \\ up_2 & \text{if } p_1/p_2 > 1 \end{cases}$$

Q4. **(30 pts)** Suppose there is a perfectly competitive industry with identical firms, each using the production function $f(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}$. Let p be the output price and let $w_K = w_L = 1$ be the input prices. Let q be the output quantity.

(a) **(5 pts)** In the short run, suppose that capital K is fixed at a constant $\bar{K}$. Find the short-run profit and output supply function.

The profit maximization problem is:

$$\max_L \; p\bar{K}^{\frac{1}{3}} L^{\frac{2}{3}} - \bar{K} - L$$

The first-order condition is:

$$\frac{\partial}{\partial L} = \frac{2}{3}p\bar{K}^{\frac{1}{3}}L^{-\frac{1}{3}} - 1 = 0$$

$$L^{\frac{1}{3}} = \frac{2}{3}p\bar{K}^{\frac{1}{3}} \qquad \rightarrow \qquad L^* = \left(\frac{2}{3}p\right)^3 \bar{K}$$

Short-run output supply is

$$q^* = \bar{K}^{\frac{1}{3}}(L^*)^{\frac{2}{3}} = \bar{K}^{\frac{1}{3}}\left(\left(\frac{2}{3}p\right)^3 \bar{K}\right)^{\frac{2}{3}}$$

$$= \bar{K}\left(\frac{2}{3}p\right)^2 = \frac{4}{9}\bar{K}p^2$$

The short-run profit function is

$$pq^* - \bar{K} - L^* = \frac{4}{9}\bar{K}p^3 - \bar{K} - \left(\frac{2}{3}p\right)^3 \bar{K} = \bar{K}\left(\frac{4}{27}p^3 - 1\right)$$

(b) **(10 pts)** Find the long-run profit function and output supply function.

The optimality conditions for the relative amounts of K and L are:

$$\frac{\frac{\partial f}{\partial K}}{\frac{\partial f}{\partial L}} = \frac{\frac{1}{3}pK^{\frac{-2}{3}}L^{\frac{2}{3}}}{\frac{2}{3}pK^{\frac{1}{3}}L^{\frac{-1}{3}}} = \frac{1}{2}\frac{L}{K} = 1$$

which gives $2K = L$. Plugging this back into the profit function, we get

$$\pi(p) = \max_K \; pK^{\frac{1}{3}}(2K)^{\frac{2}{3}} - K - 2K = \max_K 2^{\frac{2}{3}}pK - 3K = \max_K K(2^{\frac{2}{3}}p - 3)$$

This is a linear function of K, which indicates that there is no maximizer (one can always choose a higher K to get a higher profit). If $2^{\frac{2}{3}}p \leq 3$, the optimal choice of output is zero, generating zero profits. If $2^{\frac{2}{3}}p > 3$, the optimal choice of output and the optimal profit level do not exist.

Now, we consider the consumers. Suppose there are I consumers in the economy, each with an income of y. Each consumer has the following indirect utility function:

$$v(p, y) = 5\ln(y) - 2\ln(p)$$

(c) **(5 pts)** Find the consumer's Marshallian demand function.

Applying Roy's Identity, we get:

$$x(p, y) = -\frac{\frac{\partial v}{\partial p}}{\frac{\partial v}{\partial y}} = -\frac{-2/p}{5/y} = \frac{2y}{5p}$$

(d) **(10 pts)** Assume there are n firms. In the short run, find aggregate demand, aggregate supply, and the market equilibrium price.

Aggregate demand is $q^d(p) = I\frac{2y}{5p}$. Aggregate supply is $q^s(p) = n\frac{4}{9}\bar{K}p^2$. The equilibrium price is given by

$$I\frac{2y}{5p} = n\frac{4}{9}\bar{K}p^2 \qquad \rightarrow \qquad p^3 = \frac{9Iy}{10n\bar{K}}$$

$$p^* = \left(\frac{9Iy}{10n\bar{K}}\right)^{\frac{1}{3}}$$