Advanced Microeconomic Analysis

Solutions to Midterm Exam

Q1. **(20 pts)** An individual consumes two goods x_1, x_2 and his utility function is:

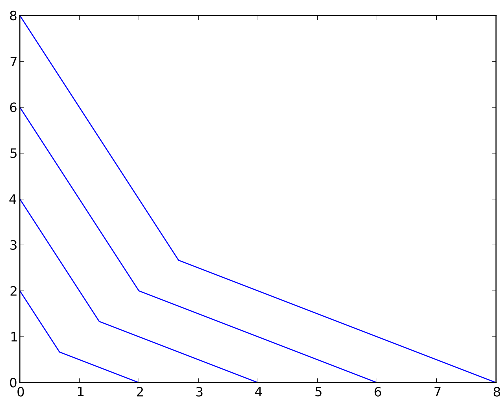
$$u(x_1, x_2) = [\min(2x_1 + x_2, x_1 + 2x_2)]^2$$

(a) Draw some indifference curves of this individual.

The indifference curves will be the same as those of $\min(2x_1 + x_2, x_1 + 2x_2)$. The first argument of the min is smaller when

$$2x_1 + x_2 < x_1 + 2x_2 \Rightarrow x_1 < x_2$$

Let's find the indifference curve for the utility level $\overline{u}^2$. If $x_1 < x_2$, then $\overline{u} = 2x_1 + x_2$ $\Rightarrow x_2 = \overline{u} - 2x_1$, with a slope of 2. If $x_1 > x_2$, then $\overline{u} = x_1 + 2x_2 \Rightarrow x_2 = \overline{u}/2 - x_1/2$, with a slope of 1/2. Therefore, the indifference curves look like this:



(b) Does the following property hold for this utility function? (You don't have to give an explanation).

- concave

  Consider the utility level along the 45-degree line, that is, $u(x, x)$ as x increases. It is equal to $9x^2$, which is not concave. Therefore, this utility function is not concave.

- quasiconcave

  As we can see from the indifference curves, the upper level sets are convex. Therefore, $u(\cdot)$ is quasiconcave.

- homogeneous

$$u(tx_1, tx_2) = [\min(2tx_1 + tx_2, tx_1 + 2tx_2)]^2$$

$$= [t \min(2x_1 + x_2, x_1 + 2x_2)]^2$$
$$= t^2 [\min(2x_1 + x_2, x_1 + 2x_2)]^2$$

Therefore, it is homogeneous of degree 2.

- homothetic

  A homothetic function can be formulated as an increasing transformation of a function that is homogeneous of degree 1. $u(x_1, x_2) = f(g(x_1, x_2))$, where $f(x) = x^2$ and $g(x_1, x_2) = \min(2x_1 + x_2, x_1 + 2x_2)$ is homogeneous of degree 1. Therefore, it is homothetic.

(c) Find the Marshallian demand for both goods.

Suppose income is y and prices of x_1, x_2 are p_1, p_2. The optimal bundle will be the point(s) where the indifference curves are tangent to the budget line with slope $-\frac{p_1}{p_2}$.

- If $\frac{p_1}{p_2} > 2$, the upper left corner is optimal: $\boldsymbol{x}^* = (0, \frac{y}{p_2})$
- If $\frac{p_1}{p_2} = 2$, all points on the upper half of the indifference curve are optimal: $\boldsymbol{x}^* = \{t(0, \frac{y}{p_2}) + (1-t)(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}) | t \in [0, 1]\}$
- If $2 > \frac{p_1}{p_2} > \frac{1}{2}$, the midpoint is optimal: $\boldsymbol{x}^* = (\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2})$
- If $\frac{p_1}{p_2} = \frac{1}{2}$, all points on the lower half of the indifference curve are optimal: $\boldsymbol{x}^* = \{t(\frac{y}{p_1}, 0) + (1-t)(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}) | t \in [0, 1]\}$
- If $\frac{p_1}{p_2} < \frac{1}{2}$, the lower right corner is optimal: $\boldsymbol{x}^* = (\frac{y}{p_1}, 0)$

(d) Find the indirect utility function.

- If $\frac{p_1}{p_2} \geq 2$, $v(p_1, p_2, y) = \left(\frac{y}{p_2)}\right)^2$
- If $2 > \frac{p_1}{p_2} > \frac{1}{2}$, $v(p_1, p_2, y) = \left(\frac{3y}{p_1+p_2}\right)^2$
- If $\frac{p_1}{p_2} \leq \frac{1}{2}$, $v(p_1, p_2, y) = \left(\frac{y}{p_1}\right)^2$

Q2. **(20 pts)** Suppose an individual consumes two goods x_1, x_2 and his indirect utility function is

$$v(p_1, p_2, y) = (y + p_1 + 2p_2) p_1^{-\frac{1}{2}} p_2^{-\frac{1}{2}}$$

(a) Find the Marshallian demand function for each of the two goods. At what levels of (p_1, p_2, y) is demand for both goods positive?

Using Roy's identity:

$$x_1(p_1, p_2, y) = \frac{-\partial v/\partial p_1}{\partial v/\partial y} = \frac{2(p_1 + 2p_2 + y) p_1^{-\frac{3}{2}} p_2^{-\frac{1}{2}} - p_1^{-\frac{1}{2}} p_2^{-\frac{1}{2}}}{p_1^{-\frac{1}{2}} p_2^{-\frac{1}{2}}}$$

$$= \frac{y - p_1 + 2p_2}{2p_1}$$

$$x_2(p_1, p_2, y) = \frac{-\partial v/\partial p_2}{\partial v/\partial y} = \frac{y + p_1 - 2p_2}{2p_2}$$

Both demands are positive when $y > 2p_2 - p_1$ and $y > p_1 - 2p_2$.

(b) Find the individual's expenditure function. Using the relation $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$:

$$e(p_1, p_2, u) = up_1^{\frac{1}{2}} p_2^{\frac{1}{2}} - p_1 - 2p_2$$

Q3. **(20 pts)** Suppose an individual lives for two periods; his "goods" are x_1, the amount consumed in period 1, and x_2, the amount consumed in period 2. His utility function over pairs (x_1, x_2) is:

$$u(x_1, x_2) = \ln(x_1) + \ln(x_2)$$

At the beginning of period 1, he has an amount of wealth $y > 0$, which can be allocated to three purposes:

- An amount $0 \le x_1 \le y$ can be consumed in period 1.
- An amount $0 \le b \le y$ can be stored; this will be available for consumption in period 2.
- An amount $0 \le z \le y$ can be invested into a risky asset; this asset's returns will be available for consumption in period 2. Suppose that with probability p, the net return is Rz, where $R > 1$; with probability $1 - p$, the net return is 0.

Therefore, the amount available for consumption in period 2 will be $b + Rz$ with probability p, and b with probability $1-p$. Find the choice of x_1, b, and z that maximizes expected utility $E\left[u(x_1, x_2)\right]$, subject to the budget constraint $x_1 + b + z \le y$.

Since ln is a strictly increasing function, the budget constraint will be satisfied with equality in both periods. In period 2, the individual will consume all available resources. The problem becomes:

$$\max_{x_1,b,z} E\left[u(x_1, x_2)\right] = \max_{x_1,b,z} \ln(x_1) + p\ln(b + Rz) + (1 - p)\ln(b) \qquad \text{s.t. } x_1 + b + z - y = 0$$

$$L(x_1, b, z, \lambda) = \ln(x_1) + p\ln(b + Rz) + (1 - p)\ln(b) - \lambda(x_1 + b + z - y)$$

$$\frac{\partial L}{\partial x_1} = \frac{1}{x_1} - \lambda = 0$$

$$\frac{\partial L}{\partial b} = \frac{p}{Rz + b} + \frac{1 - p}{b} - \lambda = 0$$

$$\frac{\partial L}{\partial z} = \frac{pR}{Rz + b} - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + b + z - y = 0$$

This has a solution

$$x_1^* = \frac{y}{2}$$

$$b^* = \frac{R - pR}{2(R-1)}y$$

$$z^* = \frac{pR - 1}{2(R-1)}y$$

Q4. **(20 pts)** Suppose there are a large number of identical firms in a perfectly competitive industry. Each firm has the long-run average cost curve:

$$AC(q) = q^2 - 12q + 50$$

where q is the firm's output.

(a) What condition must be satisfied in long-run equilibrium?

In long-run equilibrium, we assume that entry and exit of firms has driven profits to zero. Therefore, an individual firm's profit must be:

$$\pi(q) = pq - c(q) = pq - AC(q)q = 0$$

which is satisfied when $p = AC(q)$.

(b) What condition must be satisfied in a perfectly competitive industry?

In a perfectly competitive industry, we assume that firms are price-takers, and the optimal quantity is achieved when:

$$p = MC(q) = c'(Q) = \frac{d}{dq}q(q^2 - 12q + 50) = 3q^2 - 24q + 50$$

(c) Derive the long-run supply curve for this industry. In the long run, both conditions must be satisfied. Therefore, we have

$$p = AC(q) = MC(q) = q^2 - 12q + 50 = 3q^2 - 24q + 50$$

$$\Rightarrow q = 6, p = 14$$

for an individual firm. Since there are many firms, the supply curve is horizontal at $p = 14$: firms will enter the market to produce as much as is demanded.

Q5. **(20 pts)** An industry consists of many identical firms with cost function $c(q) = q^2 + 1$. When there are J active firms, each firm faces an identical inverse market demand $p = 10 - 15q - (J-1)\overline{q}$ whenever the other $J-1$ firms produce the same output level $\overline{q}$.

(a) In the short run, suppose there is no entry or exit. What is the market price and quantity in the Cournot equilibrium with J firms?

Consider firm i's profit as a function of its own quantity q_i and $\overline{q}$, which is assumed to be the output of all of the other $J-1$ firms:

$$\pi(q_i, \overline{q}) = pq_i - q_i^2 - 1 = (10 - 15q_i - (J-1)\overline{q})q_i - q_i^2 - 1$$
$$= -16q_i^2 + (10 - (J-1)\overline{q})q_i - 1$$

The optimal choice of q_i, taking $\overline{q}$ as given, satisfies the first-order condition:

$$\frac{\partial \pi}{\partial q_i} = -32q_i + 10 - (J-1)\overline{q} = 0$$

$$\Rightarrow q_i^*(\overline{q}) = \frac{10 - (J-1)\overline{q}}{32}$$

The Cournot-Nash equilibrium is when all firms individually choose their optimal quantities, taking other firms' actions as given. Therefore, each firm's quantity must satisfy:

$$q^* = \frac{10 - (J-1)q^*}{32} \Rightarrow q^* = \frac{10}{32 + J - 1}$$

Total output is $\frac{10J}{32+J-1}$. The market price is $\frac{170}{31+J}$.

(b) In the long run, entry and exit is allowed. What will be the number of active firms?

In the long run, entry and exit drives profits down to zero. Equilibrium profits with J firms is:

$$-16\left(\frac{10}{32+J-1}\right)^2 + (10 - (J-1)\frac{10}{32+J-1})\frac{10}{32+J-1} - 1 = 0$$

$$1600 = (32 + J - 1)^2 \Rightarrow J = 9$$