

Advanced Microeconomic Analysis, Lecture 1

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Welcome to Advanced Microeconomic Analysis

- ▶ This course is an introduction to the foundations of microeconomic theory, that is, the analysis of the behavior of individual agents (consumers, firms, etc)
- ▶ The first half of the course will be taught entirely in English.
- ▶ Website: <http://rncarpio.com/teaching/AdvMicro>
- ▶ Announcements, slides, & homeworks will be posted on website

About Me: Ronaldo Carpio

- ▶ BS Electrical Engineering & CS, UC Berkeley
- ▶ Master's in Public Policy, UC Berkeley
- ▶ PhD Economics, UC Davis
- ▶ Joined School of Banking & Finance in 2012

- ▶ Main textbook: *Advanced Microeconomic Theory, 3rd Ed* (2003) by Jehle & Reny
- ▶ For the first half of the course, we will roughly follow Chapters 1, 2, 3, and 7
- ▶ For those interested, a more mathematically rigorous textbook is *Microeconomic Theory* by Mas-Colell, Whinston, and Green

- ▶ There will be (tentatively) 3 problem sets, worth 10% of the grade for the first half
- ▶ A midterm exam on 11.4, worth 90% of the grade for the first half

Contacting Me

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Course Outline

- ▶ Convexity and Optimization Theory
- ▶ Consumer Theory (Chapter 1)
- ▶ Topics in Consumer Theory (Chapter 2)
- ▶ Theory of the Firm (Chapter 3)
- ▶ Game Theory (Chapter 7)
- ▶ Partial Equilibrium (Chapter 4), if time permits

Convexity

- ▶ Let's begin by considering sets of points in \mathbb{R}^n .
- ▶ A set $S \subset \mathbb{R}^n$ is *convex* if: for any two points $\mathbf{x}^1, \mathbf{x}^2 \in S$, all points on the line segment joining \mathbf{x}^1 and \mathbf{x}^2 are also in S :

$$t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in S \quad \text{for all } t \in [0, 1]$$

- ▶ The weighted average $t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, where the weights are positive and add up to 1, is called a *convex combination*.
- ▶ We can also have convex combinations of any number of points: $t_1\mathbf{x}^1 + t_2\mathbf{x}^2 + \dots + t_l\mathbf{x}^l$, where $\sum_i t_i = 1$ and $t_i \geq 0$ for all i .
- ▶ An *extreme point* of S is a point that cannot be written as a convex combination of other points in S .

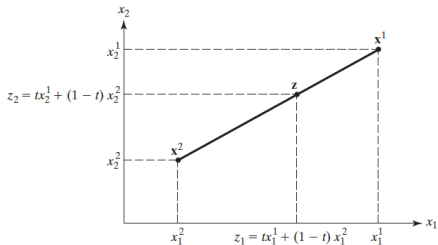


Figure A1.4. Some convex combinations in \mathbb{R}^2 .

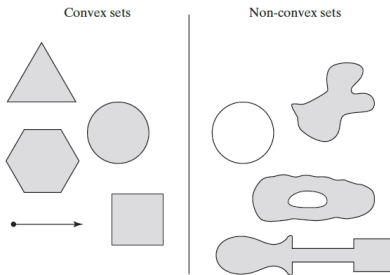


Figure A1.5. Convex and non-convex sets in \mathbb{R}^2 .

Properties of Convex Sets

- ▶ If S is convex, then the convex combination of any finite number of points in S is also in S .
- ▶ The intersection of any number of convex sets is convex.

Hyperplanes

- ▶ A *hyperplane* is a generalization of a line in 2D to N dimensions:

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} = b\}$$

- ▶ Here, \mathbf{a} is a *normal vector* to the hyperplane.
- ▶ For example, in 2D, the equation $ax + by = c$ defines a hyperplane, and (a, b) is a normal vector to this hyperplane.
- ▶ An equivalent equation defining a hyperplane with normal vector *vector*, passing through a point \mathbf{x}^0 is: $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}^0) = 0$.
- ▶ A hyperplane defines two *half-spaces*:
 - ▶ the half-space *above* the hyperplane, $\{\mathbf{x} \mid \mathbf{a} \cdot \mathbf{x} \geq b\}$
 - ▶ the half-space *below* the hyperplane, $\{\mathbf{x} \mid \mathbf{a} \cdot \mathbf{x} \leq b\}$
- ▶ In economics, the most common hyperplane is the *budget line*: if there are n goods with prices $p_1 \dots p_n$ and wealth w , then the equation $p_1 x_1 + \dots p_n x_n = w$ defines a hyperplane.

Half-Spaces

- ▶ A half-space is a convex set.
- ▶ Therefore, the (non-empty) intersection of any number of half-spaces is also a convex set.
- ▶ In the standard consumer problem with two goods, the feasible set is the intersection of three half-spaces (and therefore convex):

$$x_1 \geq 0, x_2 \geq 0, p_1x_1 + p_2x_2 \leq w$$

- ▶ A polygon with n faces is the intersection of n half-spaces.

Separating Hyperplane

- ▶ Suppose X and Y are closed, nonempty, and disjoint convex subsets of \mathbb{R}^n . If either X or Y are compact (i.e. closed and bounded), then there exists a hyperplane that *separates* X and Y .
- ▶ That is, there exists a nonzero vector $\mathbf{a} \in \mathbb{R}^n$ and scalar b such that:
 - ▶ $\mathbf{a} \cdot \mathbf{x} > b$ for all $x \in X$
 - ▶ $\mathbf{a} \cdot \mathbf{y} < b$ for all $y \in Y$
- ▶ This theorem is be used to prove the existence of general equilibrium and existence of solutions in game theory.

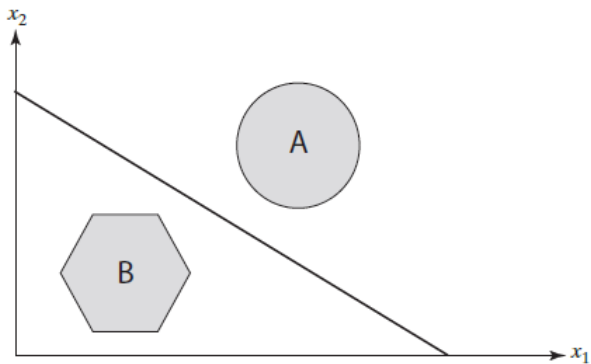


Figure A2.14. Separating convex sets.

Convex and Concave Functions

- ▶ Consider a function $f : X \rightarrow \mathbb{R}$, where X is a convex set.
- ▶ Let $\mathbf{x}^1, \mathbf{x}^2$ be any two points in X , and let \mathbf{x}^t denote their convex combination: $\mathbf{x}^t = t\mathbf{x}^1 + (1 - t)\mathbf{x}^2$, for $t \in [0, 1]$.
- ▶ f is *concave* if $f(\mathbf{x}^t) \geq tf(\mathbf{x}^1) + (1 - t)f(\mathbf{x}^2)$, for all $t \in [0, 1]$.
- ▶ f is *strictly concave* if the inequality is strict.
- ▶ f is *convex* if $f(\mathbf{x}^t) \leq tf(\mathbf{x}^1) + (1 - t)f(\mathbf{x}^2)$, for all $t \in [0, 1]$.
- ▶ f is *strictly convex* if the inequality is strict.

Epigraphs and Hypographs of a Function

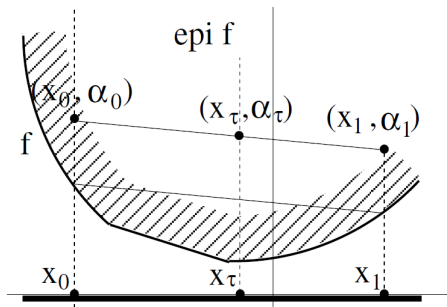
- ▶ The *epigraph* of a function f is the set of all points that are *above* the graph:

$$\{(\mathbf{x}, r) \in \mathbb{R}^{n+1} \mid r \geq f(\mathbf{x})\}$$

- ▶ Similarly, the *hypograph* of a function f is the set of all points that are *below* the graph:

$$\{(\mathbf{x}, r) \in \mathbb{R}^{n+1} \mid r \leq f(\mathbf{x})\}$$

Epigraph



Convex and Concave Functions

- ▶ A function f is concave iff its hypograph is convex.
- ▶ A function f is convex iff its epigraph is convex.

Calculus Conditions for Convexity and Concavity

- ▶ A twice-differentiable function f is concave iff its Hessian matrix (i.e. the matrix of second derivatives) is negative semidefinite at all points of its domain.
- ▶ f is strictly concave if its Hessian is negative definite at all points on its domain.
- ▶ f is (strictly) convex if its Hessian is (positive definite) positive semidefinite at all points on its domain.

Quasiconvex and Quasiconcave Functions

- ▶ The upper level set of a function f at a value r is the set $\{\mathbf{x} \in X | f(\mathbf{x}) \geq r\}$.
- ▶ Similarly, the lower level set is $\{\mathbf{x} \in X | f(\mathbf{x}) \leq r\}$.
- ▶ A function f is *quasiconcave* if its upper level sets are convex for every $r \in \mathbb{R}$.
- ▶ Similarly, f is *quasiconvex* if its lower level sets are convex for every $r \in \mathbb{R}$.
- ▶ Suppose that $f : X \rightarrow \mathbb{R}$ is quasiconcave and X is convex. The the set of maximizers of f over X is convex.

Unconstrained Optimization

- ▶ Suppose $f : X \rightarrow \mathbb{R}$ is twice continuously differentiable.
- ▶ Necessary conditions for $f(\mathbf{x})$ to be a local maximum are:
 - ▶ First-order: $\nabla f(\mathbf{x}) = 0$
 - ▶ Second-order: The Hessian of f at \mathbf{x} is positive semidefinite.
- ▶ If f is concave, then all local maxima are also global.
- ▶ If f is strictly concave, then there is a unique maximum.

Constrained Optimization

- Suppose we want to solve a maximization problem with a single *equality* constraint:

$$\max_{x_1, x_2} f(x_1, x_2) \quad \text{subject to } g(x_1, x_2) = 0$$

- For example, the standard consumer problem with 2 goods is

$$\max_{x_1, x_2} u(x_1, x_2) \quad \text{subject to } p_1 x_1 + p_2 x_2 - w = 0$$

- Lagrange's method is to form a new function, the Lagrangian, with an additional variable λ :

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$

- Then, we find an inflection point of this function, by setting its gradient to 0.

$$\nabla L = \begin{pmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \\ \frac{\partial L}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \frac{\partial f(x_1, x_2)}{\partial x_1} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_1} \\ \frac{\partial f(x_1, x_2)}{\partial x_2} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_2} \\ g(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Why does the Lagrange method work?

- ▶ Recall that the gradient of a function is the direction of *steepest increase*.
- ▶ Suppose we have a smooth function f , with gradient $\nabla f(\mathbf{x})$.
- ▶ At point \mathbf{x} , if we move at a direction not perpendicular to $\nabla f(\mathbf{x})$, the value of f changes.
- ▶ If we move in a direction perpendicular to $\nabla f(\mathbf{x})$, the value of f is constant.
- ▶ Suppose \mathbf{x} satisfies the constraint $g(\mathbf{x}) = 0$. If we are at a constrained maximum, it is not possible to move along the constraint while increasing f .
- ▶ This is only possible if $\nabla f(\mathbf{x})$ is perpendicular to g , i.e. if $\nabla f(\mathbf{x})$ is parallel to $\nabla g(\mathbf{x})$.
- ▶ This is an alternative way of writing the Lagrangian:

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}), \quad g(\mathbf{x}) = 0$$