

Advanced Microeconomic Analysis, Lecture 2

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Homework #1

- ▶ Homework #1 is posted on the course website:
<http://rncarpio.com/teaching/AdvMicro>
- ▶ Due next week at the end of lecture.
- ▶ This week, we will continue through Chapter 1 of Jehle & Reny.

Review of Last Lecture

- ▶ A set S is *convex* if any convex combination of points in S , also lies in S .
- ▶ A function f is *concave* if:
 - ▶ Its hypograph (i.e. the set of points below the graph) is convex.
 - ▶ Assuming twice differentiable, its Hessian (i.e. matrix of second derivatives) is negative semidefinite.
- ▶ A function f is *quasiconcave* if its upper level sets, i.e. $\{x | f(x) \geq r\}$ are convex, for all r .
- ▶ Constrained optimization:
 - ▶ For problems with *equality constraints*, we can form the Lagrangian and find its critical point:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$$

- ▶ The multiplier λ is nonzero.
- ▶ For problems with *inequality constraints*, some constraints will be *binding* (i.e. satisfied with equality) and the others will be *nonbinding*.

- ▶ The consumer's problem is to choose a consumption bundle, out of all feasible bundles, that is most preferred.
- ▶ There are four building blocks of the consumer's problem:
 - ▶ the *consumption set*
 - ▶ the *feasible set*
 - ▶ the *preference relation*
 - ▶ the *behavioral assumption*

Consumption & Feasible Set

- ▶ The consumption set X is the set of all possible consumption bundles.
- ▶ We will assume it is \mathbb{R}_+^n , the set of all possible non-negative quantities of n goods.
- ▶ The feasible set B is a subset of X , representing the set of consumption bundles achievable with the consumer's resources (i.e. wealth).

Preferences

- ▶ The consumer's preferences determine which bundles are more preferred.
- ▶ We will characterize preferences *axiomatically*: we will state the minimum assumptions which any reasonable consumer preference must satisfy.
- ▶ Let x^1, x^2 be any two elements of the consumption set X . We say $x^1 \succeq x^2$ if x^1 is at least as good as x^2 .

Axioms of Consumer Choice

- ▶ Axiom 1: Completeness. For all $x^1, x^2 \in X$, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$.
- ▶ Axiom 1 implies that there are no noncomparabilities, i.e. bundles that cannot be compared to each other.
- ▶ Axiom 2: Transitivity: for any three elements x^1, x^2, x^3 , if $x^1 \succsim x^2$ and $x^2 \succsim x^3$, then $x^1 \succsim x^3$.
- ▶ Axiom 2 implies that preferences are *consistent*. If preferences were not consistent, it would be possible to extract an infinite amount of money from a consumer through a *money pump*.

Strict Preference

- ▶ A binary relation \succsim is called a *preference relation* if it satisfies the axioms of completeness and transitivity.
- ▶ We can define another binary relation, $>$, defined as:

$$x^1 > x^2 \text{ iff } x^1 \succsim x^2 \text{ and } x^2 \not\succeq x^1$$

- ▶ This is called the *strict preference relation* induced by \succsim .
- ▶ If $x^1 > x^2$, we say x^1 is *strictly preferred* to x^2 .

Indifference Relation

- ▶ The binary relation \sim is defined as:

$$x^1 \sim x^2 \text{ iff } x^1 \succeq x^2 \text{ and } x^2 \succeq x^1$$

- ▶ We say that x^1 is *indifferent* to x^2 .

Sets Derived from \succsim

- ▶ Given a preference relation \succsim , we can define sets relative to a given point \mathbf{x}^0 .

- ▶ Define the set

$$\succsim(\mathbf{x}^0) = \{\mathbf{x} | \mathbf{x} \succsim \mathbf{x}^0\}$$

- ▶ This is the set of all bundles that are *at least as good as* \mathbf{x}^0 .
- ▶ Similarly, we can define $\preceq(\mathbf{x}^0)$, $<(\mathbf{x}^0)$, $>(\mathbf{x}^0)$, $\sim(\mathbf{x}^0)$.

Continuity, Local Non-satiation

- ▶ Axiom 3: Continuity. For all $\mathbf{x} \in \mathbb{R}_+^n$, $\succeq(\mathbf{x})$ and $\preceq(\mathbf{x})$ are *closed* (i.e. contains its boundary).
- ▶ The next two axioms are alternatives:
- ▶ Axiom 4': Local non-satiation. For all $\mathbf{x}^0 \in \mathbb{R}_+^n$ and for all $\epsilon > 0$, there exists some $\mathbf{x} \in B_\epsilon(\mathbf{x}^0)$ such that $\mathbf{x} \succ \mathbf{x}^0$.
 - ▶ $B_\epsilon(\mathbf{x}^0)$ is a ball of radius ϵ centered on \mathbf{x}^0 .
 - ▶ This axiom says that from any point \mathbf{x}^0 , you can always find a path that leads to a strictly more preferred bundle.
 - ▶ This rules out local maxima.

Strict Monotonicity

- ▶ Axiom 4: Strict monotonicity. For all $\mathbf{x}^0, \mathbf{x}^1$:
 - ▶ If $\mathbf{x}^0 \geq \mathbf{x}^1$ (i.e. every component of \mathbf{x}^0 is at least as large as in \mathbf{x}^1), then $\mathbf{x}^0 \succeq \mathbf{x}^1$.
 - ▶ If $\mathbf{x}^0 \gg \mathbf{x}^1$ (i.e. every component of \mathbf{x}^0 is at least as large as in \mathbf{x}^1 , and there is one strictly greater), then $\mathbf{x}^0 > \mathbf{x}^1$.
- ▶ This is a stricter condition than local non-satiation: it says that from any point \mathbf{x}^0 , any path in a direction that increases a component of \mathbf{x} leads to a strictly more preferred bundle.
- ▶ In the consumer problem, Axioms 4 and 4' guarantee that the chosen bundle will lie on the budget line (therefore, we can use the Lagrange method with an equality condition).

Convexity and Strict Convexity

- ▶ Axiom 5': Convexity. If $\mathbf{x}^1 \succeq \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succeq \mathbf{x}^0$ for all $t \in [0, 1]$.
- ▶ Axiom 5: Strict Convexity: If $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^1 \succeq \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 > \mathbf{x}^0$ for all $t \in [0, 1]$.
- ▶ Axiom 5' implies that the set $\succeq(\mathbf{x}^0)$ is convex for any \mathbf{x}^0 . Axiom 5 implies that it is *strictly* convex.
- ▶ We shall see later that this implies the utility function is (strictly) quasiconcave if (strict) convexity is satisfied.
- ▶ For the rest of this course, unless specified, we will usually assume Axiom 4 (strict monotonicity) and 5 (strict convexity).

The Utility Function

- ▶ We can summarize preferences with a *utility function* $u(\cdot)$, that assigns a number to every consumption bundle \mathbf{x} .
- ▶ $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$ iff $\mathbf{x}^0 \succeq \mathbf{x}^1$.
- ▶ $u(\mathbf{x}^0) > u(\mathbf{x}^1)$ iff $\mathbf{x}^0 \succ \mathbf{x}^1$.
- ▶ **Theorem 1.1:** If \succeq is complete, transitive, continuous and strictly monotonic, there exists a continuous real-valued function, $u: \mathbb{R}_+^n \rightarrow \mathbb{R}$ which represents \succeq .

Invariance to Positive Monotonic Transforms

- ▶ The utility function $u(\cdot)$ representing \succsim is not unique; there are infinitely many possibilities.
- ▶ Suppose we have a strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$, and let $v(\mathbf{x}) = f(u(\mathbf{x}))$. Then $v(\cdot)$ is also a utility function that represents \succsim , and vice versa.

Properties of Preferences and Utility Functions

- ▶ Suppose \succsim is represented by $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$. Then:
- ▶ 1) $u(\mathbf{x})$ is strictly increasing iff \succsim is strictly monotonic.
- ▶ 2) $u(\mathbf{x})$ is quasiconcave iff \succsim is convex.
- ▶ 3) $u(\mathbf{x})$ is strictly quasiconcave iff \succsim is strictly convex.
- ▶ As we will see, strict quasiconcavity guarantees that the consumer problem has a *unique* solution (most preferred bundle).

Consumer's problem

- ▶ Consumer's problem is to choose a bundle $\mathbf{x} = (x_1 \dots x_n)$ that is *most preferred*, from the feasible set B

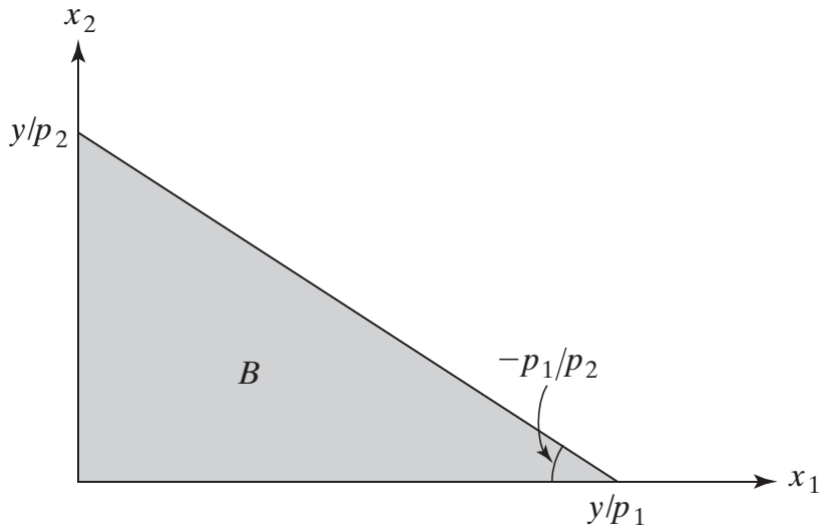
- ▶ Choose:

$$\mathbf{x}^* \in B \text{ such that } \mathbf{x}^* \succeq \mathbf{x}, \text{ for all } \mathbf{x} \in B$$

- ▶ Assume the consumer is in a *market economy*, in which all transactions occur in markets
- ▶ For all goods $i = 1 \dots n$, there is a *market price* $p_i > 0$
- ▶ Assume the consumer has no *market power*, i.e. cannot affect market price through buying and selling
- ▶ Assume the consumer has an exogenously given amount of money, y .
- ▶ Total expenditures, $\sum_{i=1}^n p_i x_i$ must be less than or equal to y .
- ▶ Then, given prices $\mathbf{p} = (p_1 \dots p_n)$, the feasible set (or *budget set*) is:

$$B = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^n p_i x_i \leq y \}$$

Budget set for 2 goods



Consumer's problem with utility functions

- ▶ Assume that preferences can be represented by a utility function $u(\mathbf{x})$, and that $u(\mathbf{x})$ is strictly quasiconcave.
- ▶ Consumer's problem becomes:

$$\max_{\mathbf{x} \in \mathbb{R}_+^n} u(\mathbf{x}) \text{ subject to } \mathbf{p} \cdot \mathbf{x} \leq y$$

- ▶ What do we know about the solution \mathbf{x}^* ?
 - ▶ A maximum *exists*, because B is closed and bounded.
 - ▶ The maximizer \mathbf{x}^* is *unique*, because $u(\mathbf{x})$ is strictly quasiconcave.
 - ▶ \mathbf{x}^* satisfies the budget constraint *with equality*, i.e. $\mathbf{p} \cdot \mathbf{x} = y$, because $u(\mathbf{x})$ is strictly monotonic.
- ▶ It would be convenient if we could guarantee all components x_i^* of \mathbf{x}^* were strictly positive (i.e. an interior solution).
- ▶ Choosing a specific form for $u(\mathbf{x})$ can ensure this.
- ▶ The solution $x_i^*(p_1, \dots, p_n, y)$, as a function of \mathbf{p} and y , is called the *Marshallian demand function*.

Solving the consumer's problem with calculus methods

- ▶ If we make the further assumption that $u(\mathbf{x})$ is *differentiable* (i.e. smooth), we can use the Lagrangian method.
- ▶ The constraint $g(x_1, \dots, x_n) = 0$ becomes: $p_1x_1 + \dots + p_nx_n - y = 0$

$$L(x_1, \dots, x_n, \lambda) = u(x_1, \dots, x_n) - \lambda(p_1x_1 + \dots + p_nx_n - y)$$

- ▶ Assume that the solution \mathbf{x}^* is strictly positive. Then the first-order conditions (called the *Kuhn-Tucker* conditions) are satisfied with some $\lambda^* > 0$:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(\mathbf{x}^*)}{\partial x_1} - \lambda^* p_1 = 0$$

⋮

$$\frac{\partial L}{\partial x_n} = \frac{\partial u(\mathbf{x}^*)}{\partial x_n} - \lambda^* p_n = 0$$

$$p_1x_1 + \dots + p_nx_n - y = 0$$

Marginal rate of substitution = price ratio

- ▶ Rearrange first-order conditions:

$$\frac{\partial u(x^*)}{\partial x_1} = \lambda^* p_1$$

⋮

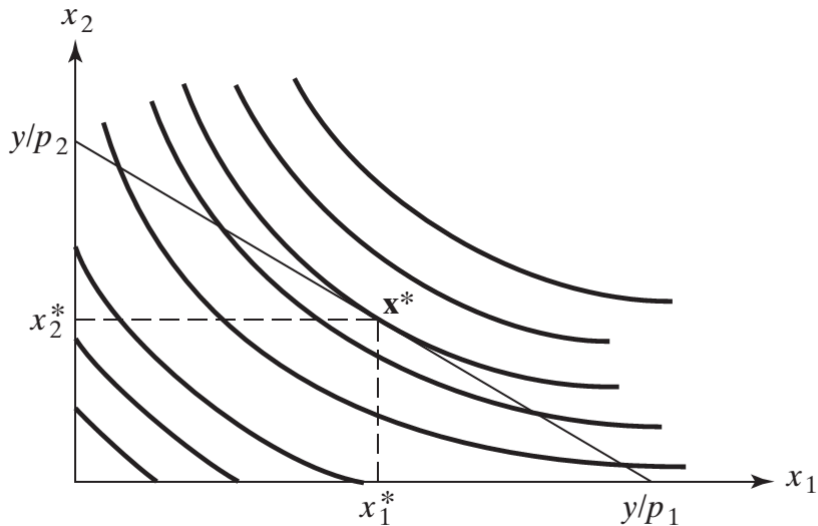
$$\frac{\partial u(x^*)}{\partial x_n} = \lambda^* p_n$$

- ▶ Assume that $\frac{\partial u(x^*)}{\partial x_1} > 0$, i.e. utility is always increasing in each good. Then for any two goods j, k :

$$\frac{\frac{\partial u(x^*)}{\partial x_j}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_j}{p_k}$$

- ▶ At the optimum, the marginal rate of substitution between any two goods equals the ratio of prices.

Solution for 2 goods



Marshallian demand function: 2 goods

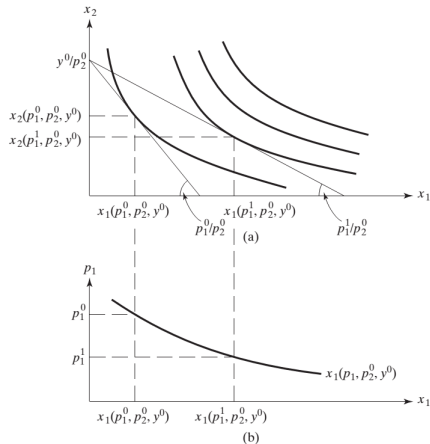


Figure 1.11. The consumer's problem and consumer demand behaviour.

Example: Constant Elasticity of Substitution (CES) utility

- ▶ For $n = 2$ and a constant $\rho < 1, \rho \neq 0$, assume utility is of the form:

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

- ▶ Is this strictly monotonic? Check that the gradient is always strictly positive for $x_1, x_2 \geq 0$:

$$\frac{\partial u}{\partial x_1} = ((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1})$$

$$\frac{\partial u}{\partial x_2} = ((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1})$$

- ▶ Note that this goes to infinity as $x_i \rightarrow 0$. This ensures that a maximizer will not have $x_1 = 0$ or $x_2 = 0$.
- ▶ Is this strictly quasiconcave? We'll mention 2 ways of checking:

Hessian of the Utility Function

- ▶ The Hessian of $u(x_1, x_2)$ is:

$$H(x_1, x_2) = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} \end{pmatrix}$$

- ▶ If $H(x_1, x_2)$ is negative definite for all $x_1 \geq 0, x_2 \geq 0$, then $u(x_1, x_2)$ is strictly concave.
- ▶ $H(x_1, x_2)$ is negative semidefinite for all $x_1 \geq 0, x_2 \geq 0$, iff $u(x_1, x_2)$ is concave.
- ▶ A matrix is negative definite if the determinants of its *upper left* $1 \times 1, 2 \times 2, \dots$ sub-matrices are negative.
- ▶ To determine if u is quasiconcave, we construct the *bordered Hessian*.
- ▶ This method works on any function. However, it's a bit tedious to calculate the determinants.

Composition of Concave Functions

- ▶ We can use these properties of concave and quasiconcave functions:
- ▶ A (strictly) concave, monotonic transformation of a (strictly) concave function is (strictly) concave.
- ▶ A monotonic transformation of a concave function is quasiconcave (not necessarily concave).
- ▶ A monotonic transformation of a quasiconcave function is quasiconcave.
- ▶ $0 < \rho < 1$, let $f(x_1, x_2) = x_1^\rho + x_2^\rho$, and let $g(z) = z^{\frac{1}{\rho}}$.
- ▶ For $\rho < 0$, let $f(x_1, x_2) = -(x_1^\rho + x_2^\rho)$, and let $g(z) = (-z)^{\frac{1}{\rho}}$.
- ▶ g is strictly concave and monotonic, and f is strictly concave, therefore $u = g \circ f$ is strictly concave.

CES Example: Consumer's Problem

$$\max_{x_1, x_2} (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} \text{ subject to } p_1 x_1 + p_2 x_2 - y \leq 0$$

- ▶ Since u is strictly monotonic, the budget constraint will hold with equality at the solution: $p_1 x_1 + p_2 x_2 - y = 0$.
- ▶ The Lagrangian function:

$$L(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

- ▶ As we saw above, the solution is strictly positive, so the Kuhn-Tucker conditions hold:

$$\frac{\partial L}{\partial x_1} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - y = 0$$

- ▶ Using the property that $MRS = \text{price ratio}$:

$$\frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{p_1}{p_2} \Rightarrow \frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

- ▶ Plugging into budget equation,

$$y = p_1 x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2 x_2$$

- ▶ Rearranging gives us:

$$x_2 = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} y, \quad x_1 = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} y$$

- ▶ These are the solutions to the consumer's problem for a given value of p_1, p_2, y , i.e. the Marshallian demand function.

Marshallian demand function: CES

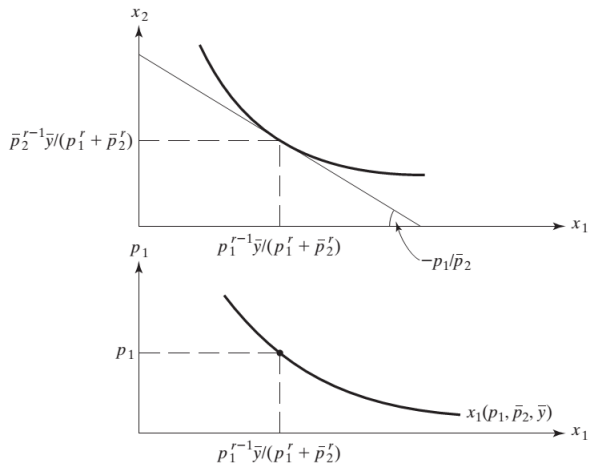


Figure 1.12. Consumer demand when preferences are represented by a CES utility function.

Conditions for a Differentiable Demand Function

- ▶ Assume that \mathbf{x}^* solves the consumer's maximization problem given \mathbf{p}^0, y^0 and is strictly positive.
- ▶ If
 - ▶ u is twice continuously differentiable on \mathbb{R}_{++}^n (i.e. \mathbf{x} strictly positive)
 - ▶ $\frac{\partial u}{\partial x_i} > 0$ for some i in $1 \dots n$
 - ▶ the bordered Hessian of u has a non-zero determinant at \mathbf{x}^* ,
- ▶ then $\mathbf{x}(\mathbf{p}, y)$ is differentiable at (\mathbf{p}^0, y^0) .

Envelope Theorem (Appendix A2.4)

- ▶ Let's go back to the general optimization problem

$$\max_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}, \mathbf{a}) \text{ subject to } g(\mathbf{x}, \mathbf{a}) = 0$$

- ▶ \mathbf{x} is the vector of choice variables, and \mathbf{a} is a vector of parameters (for consumer problems, prices and income).
- ▶ Suppose that for each \mathbf{a} , a solution exists: $\mathbf{x}^*(\mathbf{a})$ and $\lambda^*(\mathbf{a})$. The maximized value of the objective function is called the *value function*:

$$V(\mathbf{a}) = f(\mathbf{x}^*(\mathbf{a}), \mathbf{a}) = \max_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}, \mathbf{a}) \text{ s.t. } g(\mathbf{x}, \mathbf{a}) = 0$$

Envelope Theorem (Appendix A2.4)

- ▶ We would like to know how changes in \mathbf{a} affects $V(\mathbf{a})$.
- ▶ To simplify, the Envelope Theorem (A2.22 in the book) states that the effect on V is equal to the partial derivative of the Lagrangian, evaluated at the optimal values of \mathbf{x}^*, λ^* :

$$\frac{\partial V}{\partial a_j} = \frac{\partial L}{\partial a_j}(\mathbf{x}^*(\mathbf{a}), \lambda^*(\mathbf{a}))$$

Indirect Utility Function

- ▶ Going back to the consumer's problem, let $v(\mathbf{p}, y)$ denote the value function, i.e. the maximized utility function:

$$v(\mathbf{p}, y) = u(\mathbf{x}^*(\mathbf{p}, y)) = \max_{\mathbf{x}} u(\mathbf{x}) \text{ s.t. } \mathbf{p} \cdot \mathbf{x} \leq y$$

- ▶ $u(\mathbf{x})$ is called the *direct utility function*, while $v(\mathbf{p}, y)$ is called the *indirect utility function*.
- ▶ $v(\mathbf{p}, y)$ gives the highest amount of utility the consumer can reach, given prices \mathbf{p} and income y .

Indirect Utility

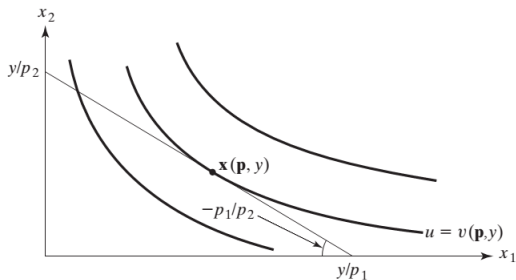


Figure 1.13. Indirect utility at prices \mathbf{p} and income y .

Properties of Indirect Utility

- ▶ If $u(\mathbf{x})$ is continuous and strictly increasing, then $v(\mathbf{p}, y)$ is:
 - ▶ Continuous
 - ▶ Homogeneous of degree zero in (\mathbf{p}, y) : for any $t > 0$, $v(t\mathbf{p}, ty) = v(\mathbf{p}, y)$
 - ▶ Strictly increasing in y
 - ▶ Decreasing in \mathbf{p} .
 - ▶ Quasiconvex in (\mathbf{p}, y)
 - ▶ Satisfies *Roy's Identity*: if $v(\mathbf{p}, y)$ is differentiable at (\mathbf{p}^0, y^0) and $\frac{\partial v}{\partial y}(\mathbf{p}^0, y^0) \neq 0$, then:

$$x_i(\mathbf{p}^0, y^0) = -\frac{\frac{\partial v}{\partial p_i}(\mathbf{p}^0, y^0)}{\frac{\partial v}{\partial y}(\mathbf{p}^0, y^0)} \text{ for } i = 1 \dots n$$

Homogeneity of Indirect Utility

- ▶ A function $f(\mathbf{x})$ is *homogeneous of degree p* if for any $t > 0$, $f(t\mathbf{x}) = t^p f(\mathbf{x})$.
- ▶ For $v(\mathbf{p}, y)$, homogeneity of degree 0 means that if both prices and income are multiplied by the same amount, the optimal consumption bundle does not change, and therefore the maximized utility does not change.
- ▶ $\mathbf{x}(\mathbf{p}, y)$ is also homogeneous of degree 0.
- ▶ There is no "money illusion": changing nominal prices and incomes does not affect the outcome.

CES Example

- ▶ Going back to the CES utility example, we found the Marshallian demand functions:

$$x_1 = \frac{p_1^{r-1}}{p_1^r + p_2^r} y$$

$$x_2 = \frac{p_2^{r-1}}{p_1^r + p_2^r} y$$

- ▶ where $r = \frac{\rho}{\rho-1}$. Note that homogeneity of degree zero holds.
- ▶ Plugging back into the direct utility function u :

$$\begin{aligned} v(p_1, p_2, y) &= \left[\left(\frac{p_1^{r-1}}{p_1^r + p_2^r} y \right)^\rho + \left(\frac{p_2^{r-1}}{p_1^r + p_2^r} y \right)^\rho \right]^{\frac{1}{\rho}} \\ &= y (p_1^r + p_2^r)^{-\frac{1}{r}} \end{aligned}$$

- ▶ This is also homogeneous of degree zero.
- ▶ We can also verify the other properties of an indirect utility function.

Expenditure Function

- ▶ The *expenditure function* is the minimum amount of expenditure necessary to achieve a given utility level u at prices \mathbf{p} :

$$e(\mathbf{p}, u) = \min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

- ▶ If preferences are strictly monotonic, then the constraint will be satisfied with equality
- ▶ Denote the solution to the expenditure minimization problem as:

$$\mathbf{x}^h(\mathbf{p}, u) = \arg \min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

- ▶ This is called the *Hicksian demand function* or *compensated demand*.
- ▶ It shows the effect of a change in prices on demand, while *holding utility constant*.

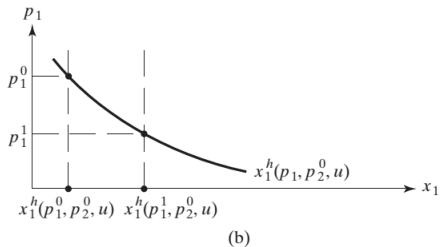
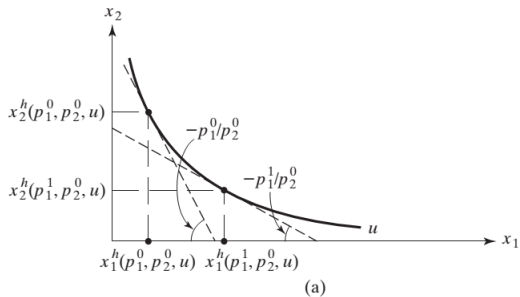


Figure 1.16. The Hicksian demand for good 1.

Properties of Expenditure Function

- ▶ If $u(\cdot)$ is continuous and strictly increasing, then $e(\mathbf{p}, u)$ is:
 - ▶ Zero when u is at the lowest possible level
 - ▶ Continuous
 - ▶ For all strictly positive \mathbf{p} , it is strictly increasing and unbounded above in u
 - ▶ Increasing in \mathbf{p}
 - ▶ Homogeneous of degree 1 in \mathbf{p}
 - ▶ Concave in \mathbf{p}
 - ▶ If $u(\cdot)$ is strictly quasiconcave, then it satisfies *Shephard's lemma*:

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0) \quad \text{for } i = 1 \dots n$$

Example: CES Utility

- ▶ Suppose direct utility is $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$, $0 \neq \rho < 1$.
- ▶ Let's derive the expenditure function:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- ▶ Form the Lagrangian:

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda((x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u)$$

- ▶ First-order conditions:

$$\frac{\partial L}{\partial x_1} = p_1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p_2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

Example: CES Utility

$$\frac{\partial L}{\partial x_1} = p_1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p_2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - u = 0$$

- ▶ Solving for x_1, x_2 , we get the Hicksian demands ($r = \frac{\rho}{\rho-1}$):

$$x_1^h(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}$$

$$x_2^h(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$$

- ▶ Plug back into objective function $\mathbf{p} \cdot \mathbf{x}$:

$$e(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}$$

Next Week

- ▶ Homework #1 is posted on the course website:
<http://rncarpio.com/teaching/AdvMicro>
- ▶ Due next week at the end of lecture.
- ▶ Please continue reading Chapter 1.