

Advanced Microeconomic Analysis, Lecture 4

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Homework #2

- ▶ Homework #2 is due next meeting.
- ▶ I will return Homework #1 next meeting.

Review of Last Week

Marshallian Demands

Homogeneity $\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty)$ for all (\mathbf{p}, y) , and $t > 0$

Symmetry $\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y) \frac{\partial x_i(\mathbf{p}, y)}{\partial y}$
 $= \frac{\partial x_j(\mathbf{p}, y)}{\partial p_i} + x_i(\mathbf{p}, y) \frac{\partial x_j(\mathbf{p}, y)}{\partial y}$ for all (\mathbf{p}, y) , and
 $i, j = 1, \dots, n$

Negative

semidefiniteness $\mathbf{z}^T \mathbf{s}(\mathbf{p}, y) \mathbf{z} \leq 0$ for all (\mathbf{p}, y) , and \mathbf{z}

Budget balancedness $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$ for all (\mathbf{p}, y) ,

Engel aggregation $\sum_{i=1}^n s_i \eta_i = 1$

Cournot aggregation $\sum_{i=1}^n s_i \varepsilon_{ij} = -s_j$ for $j = 1, \dots, n$

Hicksian Demands

Homogeneity $\mathbf{x}^h(\mathbf{p}, u) = \mathbf{x}^h(\mathbf{p}, u)$ for all (\mathbf{p}, u) , and $t > 0$

Symmetry $\frac{\partial x_i^h(\mathbf{p}, y)}{\partial p_j} = \frac{\partial x_j^h(\mathbf{p}, y)}{\partial p_i}$ for $i, j = 1, \dots, n$

Negative

semidefiniteness $\mathbf{z}^T \boldsymbol{\sigma}(\mathbf{p}, u) \mathbf{z} \leq 0$ for all \mathbf{p}, u , and \mathbf{z}

Relating the Two

Slutsky equation $\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}$ for all (\mathbf{p}, y) , $u = v(\mathbf{p}, y)$,
 $= \frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} - x_j(\mathbf{p}, y) \frac{\partial x_i(\mathbf{p}, y)}{\partial y}$ and $i, j = 1, \dots, n$

Figure 1.21. Properties of consumer demand.

Chapter 2.1: Duality

- ▶ Consider any function of prices and utility $E(\mathbf{p}, u)$ that may or may not be an expenditure function.
- ▶ Suppose E satisfies the properties of an expenditure function:
- ▶ Continuity, strictly increasing, unbounded above in u
- ▶ Increasing, homogeneous of degree 1, concave, and differentiable in \mathbf{p} .
- ▶ We can show that it is, in fact, an expenditure function for some utility function.

Constructing the Utility Function

- ▶ Choose some (\mathbf{p}^0, u^0) , evaluate $E(\mathbf{p}^0, u^0)$ at that point.
- ▶ Construct the closed half-space in the consumption set:

$$A(\mathbf{p}^0, u^0) = \{\mathbf{x} | \mathbf{p}^0 \cdot \mathbf{x} \geq E(\mathbf{p}^0, u^0)\}$$

- ▶ $A(\mathbf{p}^0, u^0)$ is a closed, convex set containing all points on or above the hyperplane defined by $\mathbf{p}^0 \cdot \mathbf{x} = E(\mathbf{p}^0, u^0)$.
- ▶ Repeat the process for all prices strictly positive prices \mathbf{p} , and take the intersection of all the half-spaces:

$$A(u^0) = \bigcap_{\mathbf{p} \gg 0} A(\mathbf{p}, u^0) = \{\mathbf{x} | \mathbf{p} \cdot \mathbf{x} \geq E(\mathbf{p}, u^0) \text{ for all } \mathbf{p} \gg 0\}$$

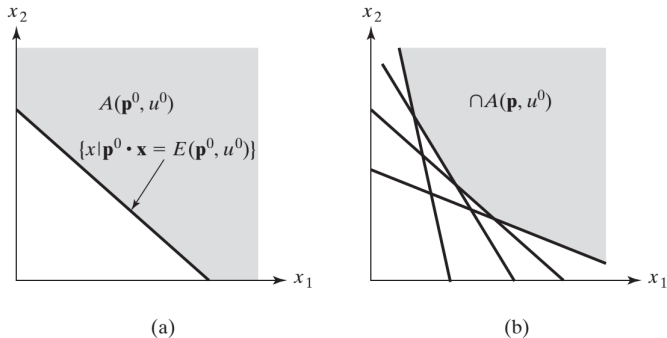


Figure 2.1. (a) The closed half-space $A(\mathbf{p}^0, u^0)$. (b) The intersection of a finite collection of the sets $A(\mathbf{p}, u^0)$.

- ▶ As the number of half-spaces increases, their intersection becomes a convex set with a smooth boundary.
- ▶ This set $A(u^0) = \bigcap_{\mathbf{p} \gg 0} A(\mathbf{p}, u^0)$ is an upper level set for some quasiconcave function.
- ▶ It turns out that this is a valid utility function.

Constructing the Utility Function

- ▶ Theorem 2.1: Let $E : \mathbb{R}_{++}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy the properties of an expenditure function. Then the function u generated by

$$u(\mathbf{x}) = \max\{u \geq 0 \mid \mathbf{x} \in A(u)\}$$

- ▶ is increasing, unbounded above, and quasiconcave.
- ▶ Theorem 2.2: The Expenditure Function of u is E :
- ▶ Let $E(\mathbf{p}, u)$ satisfy the properties of an expenditure function, and let $u(\mathbf{x})$ be derived as above. Then for all non-negative prices and utility,

$$E(\mathbf{p}, u) = \min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

Duality Between Utility and Indirect Utility

- ▶ Duality allows us to go from the expenditure function to the direct utility function.
- ▶ Since expenditure and indirect utility functions are inverses of each other, it should be possible to go from indirect to direct utility.
- ▶ **Theorem 2.3:** Suppose $u(\mathbf{x})$ is quasiconcave and differentiable on \mathbb{R}_{++}^n , with strictly positive partial derivatives. Suppose the indirect utility function generated by u is $v(\mathbf{p}, y)$. Then for all $\mathbf{x} \in \mathbb{R}_{++}^n$:

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}_{++}^n} v(\mathbf{p}, \mathbf{p} \cdot \mathbf{x})$$

- ▶ An equivalent way, which may be simpler, of obtaining u is with the problem

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}_{++}^n} v(\mathbf{p}, 1) \quad \text{subject to } \mathbf{p} \cdot \mathbf{x} = 1$$

Example 2.1: CES Utility from Indirect Utility

- Suppose we are given the indirect utility function

$$v(p_1, p_2, y) = y(p_1^r + p_2^r)^{-\frac{1}{r}}$$

- Let's find the direct utility function that generates this indirect utility. Set $y = 1$, then $v(p_1, p_2, 1) = (p_1^r + p_2^r)^{-\frac{1}{r}}$.

$$u(x_1, x_2) = \min p_1, p_2 (p_1^r + p_2^r)^{-\frac{1}{r}} \quad \text{s.t. } p_1 x_1 + p_2 x_2 - 1 = 0$$

$$L(p_1, p_2, \lambda) = (p_1^r + p_2^r)^{-\frac{1}{r}} - \lambda(p_1 x_1 + p_2 x_2 - 1)$$

Example 2.1: CES Utility from Indirect Utility

- ▶ First order conditions:

$$\frac{\partial L}{\partial p_1} = -(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1} - \lambda x_1 = 0$$

$$\frac{\partial L}{\partial p_2} = -(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1} - \lambda x_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - 1 = 0$$

- ▶ Solving this system of equations for $u(x_1, x_2)$ gives us

$$u(x_1, x_2) = (x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}})^{\frac{r-1}{r}}$$

- ▶ which is the original CES utility function, with $\rho = r/(r-1)$.

Utility Maximization and Expenditure Minimization

- ▶ There are two equivalent ways of characterizing consumer demand.
- ▶ One is to start with the direct utility function and derive Marshallian demand.
- ▶ Or, we can start with an expenditure function and use inversion and differentiation to derive demand.
- ▶ One way may be analytically simpler than the other, or may be empirically easier to observe.
- ▶ For example, we cannot directly observe utilities, but we can observe prices and expenditures.

Chapter 3: Theory of the Firm

- ▶ What is a firm? Here, a firm is an entity that:
 - ▶ acquires inputs;
 - ▶ combines them to produce outputs;
 - ▶ sells outputs on the market.
- ▶ The firm's objective is to maximize profits.

Production

- ▶ Production is the process of *transforming inputs into outputs*.
- ▶ Technological feasibility determines what is possible.
- ▶ Two common ways of representing production:
- ▶ a *production possibility set* $Y \subset \mathbb{R}^n$
 - ▶ Each vector $\mathbf{y} \in Y$ is a *production plan* whose components are the various inputs and outputs.
 - ▶ By convention, $y_i < 0$ if resource i is used up, and $y_i > 0$ if it is produced.
 - ▶ Possible to have multiple outputs, resources that are both inputs and outputs.
- ▶ a *production function*:
 - ▶ one output, $y = f(\mathbf{x})$.

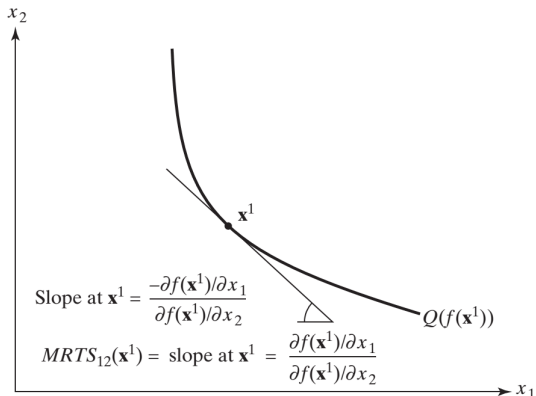
Production Functions

- ▶ We will assume that the production function $f(\mathbf{x})$ is continuous, strictly increasing, strictly quasiconcave, and $f(\mathbf{0}) = 0$.
- ▶ Here, the quasiconcave assumption implies there are *complementarities* in production, that is, production is decreased when there is an extreme amount of one input.
- ▶ The *marginal product of input i* is $\frac{\partial f(\mathbf{x})}{\partial x_i}$.
- ▶ If f is differentiable, then $\frac{\partial f(\mathbf{x})}{\partial x_i} > 0$.
- ▶ For a given level of output y , the set of input vectors producing y is called the y -level *isoquant*:

$$Q(y) = \{\mathbf{x} \geq 0 \mid f(\mathbf{x}) = y\}$$

- ▶ Similar to indifference curves of utility functions.
- ▶ For a given input vector \mathbf{x} , the *isoquant through \mathbf{x}* is $Q(f(\mathbf{x}))$, the set of input vectors producing the same output as \mathbf{x} .

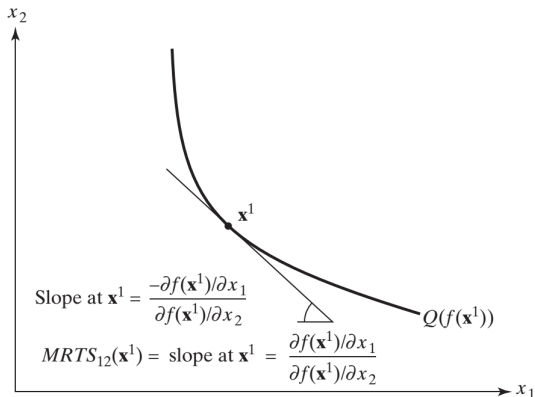
Marginal Rate of Technical Substitution



- ▶ The marginal rate of technical substitution (MRTS) is:

$$MRTS_{ij}(\mathbf{x}) = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j}$$

Marginal Rate of Technical Substitution



- ▶ MRTS is similar to MRS for consumers: the rate at which one input can be substituted for another, while keeping output level constant.
- ▶ We will denote $\frac{\partial f(x)}{\partial x_i}$ as $f_i(\mathbf{x})$.

Separable Production Functions

- ▶ In general, the MRTS at inputs \mathbf{x} depends on the level of each input.
- ▶ Frequently, we would like to divide inputs into "types". The MRTS of inputs within one type is not affected by inputs in another type.
- ▶ Suppose there are N inputs, and we can partition them into $S > 1$ mutually exclusive subsets $N_1 \dots N_S$. The production function is *weakly separable* if the MRTS between two inputs is independent of inputs used in other groups:

$$\frac{\partial(f_i(\mathbf{x})/f_j(\mathbf{x}))}{\partial x_k} = 0 \quad \text{for } i, j \in N_s, k \notin N_s$$

- ▶ The production function is *strongly separable* if the MRTS between two inputs from any two groups is independent of all inputs outside those two groups.

$$\frac{\partial(f_i(\mathbf{x})/f_j(\mathbf{x}))}{\partial x_k} = 0 \quad \text{for } i \in N_s, j \in N_t, k \notin N_s \cup N_t$$

Separable Production Functions

- ▶ For example: suppose we have two types of inputs, "capital" and "labor".
- ▶ If the MRTS between two "capital" inputs is unaffected by the level of "labor" inputs, it is weakly separable.

Elasticity of Substitution

- ▶ The elasticity of substitution of input j for input i at point \mathbf{x}^0 is:

$$\sigma_{ij}(\mathbf{x}^0) = \left(\frac{d \ln MRTS_{ij}(\mathbf{x}(r))}{d \ln r} \Big|_{r = \frac{x_j^0}{x_i^0}} \right)^{-1}$$

- ▶ $r = \frac{x_j^0}{x_i^0}$, the ratio of x_j to x_i at \mathbf{x}^0
- ▶ $\mathbf{x}(r)$ is the vector of inputs on $Q(f(\mathbf{x}^0))$ that maintains the ratio r .
- ▶ $\sigma_{ij}(\mathbf{x}^0)$ is a measure of the curvature of the isoquant through \mathbf{x}^0 .
- ▶ If f is quasiconcave, $\sigma_{ij} \geq 0$.
- ▶ The larger σ_{ij} , the easier it is to substitute between i and j .

$$\begin{aligned}\frac{d \ln \left(\frac{x_j}{x_i} \right)}{d \ln MRTS_{ij}} &= \frac{d \ln \left(\frac{x_j}{x_i} \right)}{d \ln \left(\frac{df}{dx_i} / \frac{df}{dx_j} \right)} \\ &= \frac{d(x_j/x_i)}{x_j/x_i} \frac{\frac{df}{dx_i} / \frac{df}{dx_j}}{d \left(\frac{df}{dx_i} / \frac{df}{dx_j} \right)}\end{aligned}$$

Example 3.1: CES Production

- ▶ Suppose the production function is CES:

$$y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$$

$$MRTS_{12}(x_1, x_2) = \left(\frac{x_2}{x_1}\right)^{1-\rho}$$

- ▶ Set $r = \frac{x_2}{x_1}$.

$$\begin{aligned}\frac{d \ln MRTS_{12}(x(r))}{d \ln r} &= \frac{d \ln r^{1-\rho}}{d \ln r} \\ &= (1-\rho) \frac{d \ln r}{d \ln r} = 1-\rho\end{aligned}$$

- ▶ Elasticity is a constant, hence CES.

CES & Cobb-Douglas

- ▶ CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$, $0 \neq \rho < 1$
- ▶ $\sigma = 1/(1 - \rho)$, as $\rho \rightarrow 1$, σ increases, therefore x_1, x_2 become *more* substitutable for each other.
- ▶ Consider a modified version of CES with weights α_i :

$$y = \left(\sum_{i=1}^n \alpha_i x_i^\rho \right)^{\frac{1}{\rho}}, \sum_{i=1}^n \alpha_i = 1$$

$$MRTS_{ij}(x_i, x_j) = \frac{\alpha_j}{\alpha_i} \left(\frac{x_j}{x_i} \right)^{1-\rho}$$

- ▶ As $\rho \rightarrow 0$, $MRTS_{ij} = \frac{\alpha_j}{\alpha_i} \left(\frac{x_i}{x_j} \right)$
- ▶ This is the *MRTS* of the Cobb-Douglas production function:

$$y = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

- ▶ Therefore, Cobb-Douglas is a limiting case of CES.

- ▶ CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$
- ▶ $\sigma_{ij} = 1/(1 - \rho)$ for all $i \neq j$.
- ▶ As $\rho \rightarrow -\infty, \sigma_{ij} \rightarrow 0$, there is *no* substitutability between outputs
- ▶ This is the Leontief production function:

$$y = \min(x_1, \dots, x_n)$$

- ▶ Leontief production is also a limiting case of CES.

Homogeneous Production Functions are Concave

- ▶ Theorem 3.1: Let $f(\mathbf{x})$ be a continuous, strictly increasing, strictly quasiconcave production function, and suppose that f is homogeneous of degree $0 < \alpha \leq 1$. Then $f(\mathbf{x})$ is concave.
- ▶ Proof: First, suppose $\alpha = 1$. Take any strictly positive $\mathbf{x}^1, \mathbf{x}^2$, let $y^1 = f(\mathbf{x}^1), y^2 = f(\mathbf{x}^2)$.

$$f\left(\frac{\mathbf{x}^1}{y^1}\right) = f\left(\frac{\mathbf{x}^2}{y^2}\right) = 1$$

- ▶ By strict quasiconcavity of f ,

$$f\left(\frac{t\mathbf{x}^1}{y^1} + \frac{(1-t)\mathbf{x}^2}{y^2}\right) \geq 1 \quad \text{for } 0 \leq t \leq 1$$

- ▶ Choose $t^* = \frac{y^1}{y^1+y^2}, (1-t^*) = \frac{y^2}{y^1+y^2}$.

$$f\left(\frac{\mathbf{x}^1}{y^1+y^2} + \frac{\mathbf{x}^2}{y^1+y^2}\right) \geq 1$$

Homogeneous Production Functions are Concave

$$f\left(\frac{\mathbf{x}^1}{y^1 + y^2} + \frac{\mathbf{x}^2}{y^1 + y^2}\right) \geq 1$$

- ▶ By linear homogeneity of f ,

$$f(\mathbf{x}^1 + \mathbf{x}^2) \geq y^1 + y^2 = f(\mathbf{x}^1) + f(\mathbf{x}^2)$$

- ▶ Now, consider any $0 \leq t \leq 1$, and substitute $t\mathbf{x}^1, (1-t)\mathbf{x}^2$ into the previous equation:

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \geq f(t\mathbf{x}^1) + f((1-t)\mathbf{x}^2) = tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2)$$

- ▶ which is the characterization of a concave function.

Homogeneous Production Functions are Concave

- ▶ Now, we consider the case where $f(\mathbf{x})$ is homogeneous of degree α , $0 < \alpha \leq 1$.
- ▶ The composition $(f(\mathbf{x}))^{\frac{1}{\alpha}}$ is homogeneous of degree 1, and satisfies the assumptions on a production function (continuous, strictly increasing, strictly quasiconcave, $f(0) = 0$).
- ▶ Therefore, $(f(\mathbf{x}))^{\frac{1}{\alpha}}$ is concave, as shown above.
- ▶ The further composition $\left[(f(\mathbf{x}))^{\frac{1}{\alpha}} \right]^{\alpha} = f(\mathbf{x})$ has an outer function that is concave (since $\alpha \leq 1$).
- ▶ Therefore, the whole thing is concave, so $f(\mathbf{x})$ is concave.

Returns to Scale

- ▶ We want to know how output responds as the amounts of inputs change.
- ▶ *Returns to scale* refers to how output changes when all inputs change proportionally, i.e. multiply all inputs by the same number.
- ▶ The *marginal product* of input i : $MP_i(\mathbf{x}) = f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}$
- ▶ The *average product* of input i : $AP_i(\mathbf{x}) = f(\mathbf{x})/x_i$
- ▶ The *output elasticity* of input i , i.e. the percentage change in output resulting from a 1% change in input i :

$$\mu_i(\mathbf{x}) = \frac{f_i(\mathbf{x})x_i}{f(\mathbf{x})} = \frac{MP_i(\mathbf{x})}{AP_i(\mathbf{x})}$$

Global Returns to Scale

- ▶ A production function $f(\mathbf{x})$ has the following property if the corresponding condition holds for all \mathbf{x} :
 - ▶ *Constant* returns to scale if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $t > 0$.
 - ▶ *Increasing* returns to scale if $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $t > 1$.
 - ▶ *Decreasing* returns to scale if $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $t > 1$.
- ▶ A production function with constant returns to scale must be a linear homogeneous function.
- ▶ Every homogeneous production function with degree > 1 must have increasing returns; degree < 1 must have decreasing returns.
- ▶ The converse is not necessarily true.

Local Returns to Scale

- ▶ A production function need not have the same returns to scale everywhere.
- ▶ We would like a *local* measure of returns to scale at point \mathbf{x} .
- ▶ The *elasticity of scale* or *overall elasticity of output* is:

$$\mu(\mathbf{x}) = \lim_{t \rightarrow 1} \frac{d \ln [f(t\mathbf{x})]}{d \ln(t)} = \frac{\sum_{i=1}^n f_i(\mathbf{x})x_i}{f(\mathbf{x})}$$

- ▶ Compare to the output elasticity of a single input i :

$$\mu_i(\mathbf{x}) = \frac{f_i(\mathbf{x})x_i}{f(\mathbf{x})} = \frac{MP_i(\mathbf{x})}{AP_i(\mathbf{x})}$$

- ▶ If $\mu(\mathbf{x})$ is equal/greater than/less than 1, returns to scale are locally constant/increasing/decreasing.
- ▶ Elasticity of scale and output elasticities of input i are related:

$$\mu(\mathbf{x}) = \sum_{i=1}^n \mu_i(\mathbf{x})$$

Example 3.2

- Consider this production function with variable returns to scale:

$$y = k(1 + x_1^{-\alpha} x_2^{-\beta})^{-1}$$

- where $\alpha > 0, \beta > 0$, and k is an upper bound on the level of output: y cannot exceed k .

$$f_1(\mathbf{x}) = \frac{\alpha k x_1^{\alpha-1} x_2^{\beta}}{(1 + x_1^{\alpha} x_2^{\beta})^2}, f_2(\mathbf{x}) = \frac{\beta k x_1^{\alpha} x_2^{\beta-1}}{(1 + x_1^{\alpha} x_2^{\beta})^2}$$

$$\mu_1(\mathbf{x}) = \frac{f_1(\mathbf{x})x_1}{f(\mathbf{x})} = \frac{\alpha x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}, \mu_2(\mathbf{x}) = \frac{f_2(\mathbf{x})x_2}{f(\mathbf{x})} = \frac{\beta x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}$$

$$\mu(\mathbf{x}) = \frac{(\alpha + \beta)x_1^{\alpha} x_2^{\beta}}{1 + x_1^{\alpha} x_2^{\beta}}$$

- Elasticities depend on x_1, x_2 .

Example 3.2

- ▶ We can rewrite the elasticities to be functions of the level of output $y = k(1 + x_1^{-\alpha} x_2^{-\beta})^{-1}$.

$$x_1^{-\alpha} x_2^{-\beta} = \frac{k}{y} - 1$$

$$\mu_1^*(y) = \frac{\alpha x_1^\alpha x_2^\beta}{1 + x_1^\alpha x_2^\beta} = \alpha \left(1 - \frac{y}{k}\right)$$

$$\mu_2^*(y) = \frac{\beta x_1^\alpha x_2^\beta}{1 + x_1^\alpha x_2^\beta} = \beta \left(1 - \frac{y}{k}\right)$$

$$\mu^*(y) = (\alpha + \beta) \left(1 - \frac{y}{k}\right)$$

- ▶ Returns to scale decline monotonically as output increases.
- ▶ At $y = 0$, $u^*(y) = \alpha + \beta > 0$. At $y \rightarrow k$, $u^*(y) \rightarrow 0$.
- ▶ If $\alpha + \beta > 1$, returns to scale will go from increasing to decreasing as y goes up.

Cost Function

- ▶ The firm's *cost of output* of a level y is the expenditure it must make to acquire the inputs necessary to produce y .
- ▶ If the firm's objective is to maximize profits, then for a given y , it must try to minimize costs (since revenues are determined by y).
- ▶ We will assume that firms are perfectly competitive on their input markets, and therefore are *price-takers*.
- ▶ Let $\mathbf{w} = (w_1, \dots, w_n)$ denote the market prices of inputs $1, \dots, n$.
- ▶ Let $\mathbf{x} = (x_1, \dots, x_n)$ denote the quantities of inputs $1, \dots, n$.
- ▶ The firm will choose \mathbf{x} to minimize the expenditure $\mathbf{w} \cdot \mathbf{x}$, subject to producing output $f(\mathbf{x}) \geq y$.

Cost Function

- ▶ The *cost function*, for strictly positive input prices \mathbf{w} and feasible output levels $y \in f(\mathbb{R}_+^n)$, is defined as:

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in \mathbb{R}_+^n} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y$$

- ▶ Let $\mathbf{x}(\mathbf{w}, y)$ denote the solution to this problem. Then:

$$c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{x}(\mathbf{w}, y)$$

- ▶ Since we assume that $f(\cdot)$ is strictly increasing, the constraint will always be binding at a solution (similar to: monotonicity of utility guarantees the budget constraint is satisfied with equality).

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in \mathbb{R}_+^n} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) = y$$

Cost Function

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in \mathbb{R}_+^n} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) = y$$

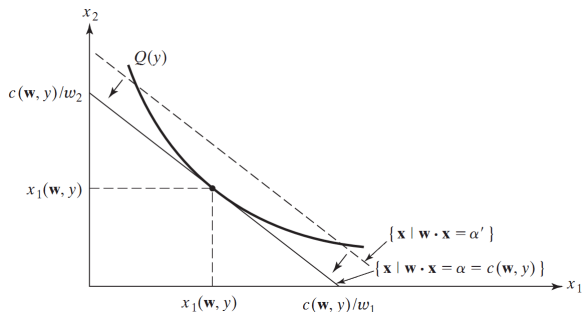
$$L(x_1, \dots, x_n, \lambda) = \mathbf{w} \cdot \mathbf{x} - \lambda(f(\mathbf{x}) - y)$$

$$\frac{\partial L}{\partial x_i} = w_i - \lambda \frac{\partial f(\mathbf{x})}{\partial x_i}$$

$$w_i = \lambda^* \frac{\partial f(\mathbf{x})}{\partial x_i} \quad \text{for } i = 1, \dots, n$$

$$MRTS_{i,j} = \frac{\partial f(\mathbf{x}^*)/\partial x_i}{\partial f(\mathbf{x}^*)/\partial x_j} = \frac{w_i}{w_j}$$

Conditional Input Demand



- ▶ The solution $\mathbf{x}^* = \mathbf{x}(\mathbf{w}, y)$ is called the firm's *conditional input demand*, i.e. demand conditional on producing at least output level y .
- ▶ The solution to the cost-minimization problem is the tangency point between the y -level isoquant, and an *isocost* line of the form $\mathbf{w} \cdot \mathbf{x} = \alpha$ for some $\alpha > 0$.

Example 3.3

- ▶ Suppose production is CES: $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$
- ▶ Cost minimization problem:

$$\min_{x_1 \geq 0, x_2 \geq 0} w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad (x_1^\rho + x_2^\rho) = y$$

- ▶ First-order conditions:

$$MRTS_{12} = \frac{w_1}{w_2} = \left(\frac{x_1}{x_2} \right)^{\rho-1}$$

$$y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = x_2 w_2^{\frac{-1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}$$

- ▶ Conditional input demands:

$$x_1 = y w_1^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{-1}{\rho}}$$

$$x_2 = y w_2^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{-1}{\rho}}$$

Example 3.3

$$x_1 = yw_1^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{-1}{\rho}}$$

$$x_2 = yw_2^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \frac{1}{\rho})^{\frac{-1}{\rho}}$$

- ▶ Plug x_1^*, x_2^* into $w_1x_1 + w_2x_2$ to get cost function:

$$c(w_1, w_2, y) = y(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}}$$

Similarities to Expenditure Function

- ▶ Obviously, this is very similar to the expenditure function in consumer theory.
- ▶ Expenditure function:

$$e(\mathbf{p}, u) = \min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

- ▶ Cost function:

$$c(\mathbf{w}, u) = \min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq u$$

- ▶ All the properties and theorems that apply to expenditure functions also apply to cost functions.

Properties of Cost Function

- ▶ Theorem 3.2: If f is continuous and strictly increasing, then $c(\mathbf{w}, y)$ has the following properties:
 - ▶ $c(\mathbf{w}, 0) = 0$
 - ▶ is continuous on its domain
 - ▶ For strictly positive \mathbf{w} , is strictly increasing and unbounded above in y
 - ▶ Increasing in \mathbf{w}
 - ▶ Homogeneous of degree 1 in \mathbf{w}
 - ▶ Concave in \mathbf{w}
- ▶ If f is strictly quasiconcave (therefore guaranteeing a unique solution), then

$$\frac{\partial c(\mathbf{w}^0, y^0)}{\partial w_i} = x_i(\mathbf{w}^0, y^0) \quad \text{for } i = 1, \dots, n$$

- ▶ The last property is known as Shephard's lemma.

Example 3.4

- ▶ Suppose the cost function is Cobb-Douglas: $c(w_1, w_2, y) = Aw_1^\alpha w_2^\beta y$
- ▶ By Shephard's lemma, conditional input demand is $\frac{\partial c}{\partial w_i}$:

$$x_1(w_1, w_2, y) = \alpha Aw_1^{\alpha-1} w_2^\beta y = \frac{\alpha c(w_1, w_2, y)}{w_1}$$

$$x_2(w_1, w_2, y) = \beta Aw_1^\alpha w_2^{\beta-1} y = \frac{\beta c(w_1, w_2, y)}{w_2}$$

- ▶ Ratio of conditional input demands:

$$\frac{x_1(w_1, w_2, y)}{x_2(w_1, w_2, y)} = \frac{\alpha w_2}{\beta w_1}$$

- ▶ With a Cobb-Douglas cost function, the ratio of inputs depends only on the ratio of input prices, independent of the level of output.

Example 3.4

- ▶ Define the *input share* of input i as: $s_i = \frac{w_i x_i(w_1, w_2, y)}{c(w_1, w_2, y)}$, i.e. the fraction of total expenditure spent on input i .
- ▶ For this cost function, $s_1 = \alpha$, $s_2 = \beta$.

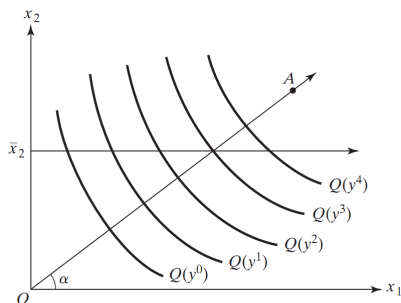
Properties of Conditional Input Demands

- ▶ These are the same as the properties of Hicksian compensated demands.
- ▶ Theorem 3.3: Suppose f is continuous, strictly increasing, and strictly quasiconcave. Then:
 - ▶ $\mathbf{x}(\mathbf{w}, y)$ is homogeneous of degree zero in \mathbf{w}
 - ▶ The substitution matrix $\sigma^*(\mathbf{w}, y)$ is symmetric and negative semidefinite:

$$\sigma^*(\mathbf{w}, y) = \begin{pmatrix} \frac{\partial x_1(\mathbf{w}, y)}{\partial w_1} & \cdots & \frac{\partial x_1(\mathbf{w}, y)}{\partial w_n} \\ \vdots & & \vdots \\ \frac{\partial x_n(\mathbf{w}, y)}{\partial w_1} & \cdots & \frac{\partial x_n(\mathbf{w}, y)}{\partial w_n} \end{pmatrix}$$

- ▶ This implies that $\frac{\partial x_i(\mathbf{w}, y)}{\partial w_i} \leq 0$ for all i . (Recall: Hicksian demand always decreases as own price increases).

Homothetic Production Functions



- ▶ A *homothetic* production function has the property that $MRTS_{ij}(x_i, x_j)$ is homogeneous of degree zero.
- ▶ For example, with CES production, $MRTS_{12}(x_1, x_2) = \left(\frac{x_2}{x_1}\right)^{1-\rho}$.
- ▶ Along rays from the origin, the slope of the isoquants are the same.

Homothetic Production Functions

- ▶ A homothetic function can be written in the form $H(f(\mathbf{x}))$, where $H(\cdot)$ is a monotonically increasing function, and $f(\mathbf{x})$ is a homogeneous function of any degree.
- ▶ Theorem 3.4: When $f(\cdot)$ is continuous, strictly increasing, strictly quasiconcave, and homothetic, then:
 - ▶ The cost function is *multiplicatively separable* in \mathbf{w}, y : there is some strictly increasing $h(y)$ such that:

$$c(\mathbf{w}, y) = h(y)c(\mathbf{w}, 1)$$

- ▶ Conditional input demands are multiplicatively separable in \mathbf{w}, y : there is some strictly increasing $h'(y)$ such that:

$$\mathbf{x}(\mathbf{w}, y) = h'(y)\mathbf{x}(\mathbf{w}, 1)$$

Homothetic Production Functions

- ▶ When $f(\cdot)$ is homogeneous of degree $\alpha > 0$:

$$c(\mathbf{w}, y) = y^{\frac{1}{\alpha}} c(\mathbf{w}, 1)$$

$$\mathbf{x}(\mathbf{w}, y) = y^{\frac{1}{\alpha}} \mathbf{x}(\mathbf{w}, 1)$$

Short-Run Costs

- ▶ So far, we have been assuming that all inputs can be changed when calculating the appropriate cost.
- ▶ This may not be true in all situations. In particular, in the *short run*, some inputs may be fixed (e.g. if "capital" inputs, such as factories or machines, take a long time to build).
- ▶ We denote the *long-run* cost function as the one where all inputs can be changed.
- ▶ The *short-run* cost function is where some inputs are fixed.

Short-Run Costs

- ▶ Let the production function be $f(\mathbf{z})$, where we divide the inputs \mathbf{z} into *variable* inputs \mathbf{x} , and *fixed* inputs $\bar{\mathbf{x}}$:

$$\mathbf{z} = (\mathbf{x}, \bar{\mathbf{x}})$$

- ▶ The *short-run*, or *restricted* cost function, is defined as:

$$sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) = \min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} + \bar{\mathbf{w}} \cdot \bar{\mathbf{x}} \quad \text{s.t. } f(\mathbf{x}, \bar{\mathbf{x}}) \geq y$$

- ▶ That is, we are taking the fixed inputs $\bar{\mathbf{x}}$ as parameters to the problem, but they still contribute to expenditures.
- ▶ If $\mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$ solves this problem, then

$$sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) = \mathbf{w} \cdot \mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) + \bar{\mathbf{w}} \cdot \bar{\mathbf{x}}$$

Short-Run Costs

- ▶ The optimized cost of the variable inputs, $\mathbf{w} \cdot \mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$, is called *total variable cost*.
- ▶ The cost of the fixed inputs, $\bar{\mathbf{w}} \cdot \bar{\mathbf{x}}$, is called *total fixed cost*.

Properties of Short-Run Costs

- ▶ For a given level of output, long-run costs (where the firm can set all input levels) cannot be greater than short-run costs (where the firm cannot set all input levels).
- ▶ This is because the feasible region for z of the short-run problem is a *subset* of the feasible region of the long-run problem.
- ▶ Any feasible z in the short-run problem is feasible in the long-run problem, but not vice versa.
- ▶ In general, if set $A \subset B$, then for any function g :

$$\max_{x \in A} g(x) \leq \max_{x \in B} g(x)$$

$$\min_{x \in A} g(x) \geq \min_{x \in B} g(x)$$

Homework #2

- ▶ Homework #2 is due next week.
- ▶ I will return Homework #1 next week.
- ▶ Please read the rest of Chapter 3, then we will move to Chapter 7.