Advanced Microeconomic Analysis, Lecture 5

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Announcements

- Homework #3 is due next week.
- The midterm will be on Nov. 4.
- Midterm will be open-book.
- Chapters 1, 2.1, 3, and 7 will be covered.

Pure Strategy Nash Equilibrium

- **Def 7.7** Given a strategic form game $G = (S_i, u_i)_{i=1}^N$, the joint strategy $\hat{s} \in S$ is a *pure strategy Nash equilibrium* of G is: for each player i,

$$u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{-i}) \qquad \text{for all } s_i \in S_i$$

- Each player cannot find an alternative action that would give him a strictly higher payoff, keeping all other players' strategies constant.

Best Response Correspondence

- Suppose we have a game $G = (S_i, u_i)_{i=1}^N$. Let s_{-i} be any joint strategy of the players except for Player i.
- Player i's *best response correspondence* $B_i(s_{-i})$ is the set of strategies of Player i that give the highest possible payoff when s_{-i} are played:

$$B_i(s_{-i}) = \{\hat{s}_i \in S_i | u_i(\hat{s}_i, s_{-i} \geq u_i(s_i, s_{-i}) \qquad \text{for all } s_i \in S_i\}$$

- A member of the set $B_i(s_{-i})$ is called a *best response* of Player i to s_{-i}.
- If $B_i(s_{-i})$ is single-valued, i.e. there is always a unique payoff-maximizing strategy of Player i in response to s_{-i}, we also call this the *best response function*.
- A joint strategy s is a Nash equilibrium if every player is playing a best response to the joint strategy of the other players.
- In 2-player games, one way to find NE is to plot each player's best response correspondence, and find the intersection.

Example: Prisoner's Dilemma

	Cooperate	*Defect*
Cooperate	2,2	0,3
Defect	3,0	1,1

- The best response correspondences of Player 1 are: $B_1(Cooperate) = \{Defect\}, B_1(Defect) = \{Defect\}$
- The best response correspondences of Player 2 are: $B_2(Cooperate) = \{Defect\}, B_2(Defect) = \{Defect\}$
- There is one intersection $(Defect, Defect)$, which is therefore the unique Nash equilibrium.

Example: A Joint Project

- Suppose 2 players are working on a joint project.
- Player i chooses the amount of work x_i to contribute, where $x_i \geq 0$.
- The payoff to each Player i is determined by the contributions of both players:

$$u_i = x_i(c + x_j - x_i)$$

- where x_j is the contribution of the other player, and $c > 0$ is a constant.
- Note that in this game, each player's strategy set is infinite: $S_i = [0, \infty)$.

Example: A Joint Project

$$u_i = x_i(c + x_j - x_i) = -x_i^2 + (c + x_j)x_i$$

- The payoff function $u_i(x_i)$ is concave in x_i, so we can use calculus to maximize it.
- Setting $u_i'(x_i) = -2x_i + c + x_j = 0$, we get Player i's best response function: $B_i(x_j) = (c + x_j)/2$.
- Player 1 and Player 2's best response functions are:

$$B_1(x_2) = (c + x_2)/2, B_2(x_1) = (c + x_1)/2$$

- At a Nash equilibrium, each player must be playing a best response to the other's strategy:

$$x_1 = (c + x_2)/2, x_2 = (c + x_1)/2$$

- Solving for x_1 and x_2, we get $x_1 = x_2 = c$.

Mixed Strategies

- Now, suppose that instead of simply choosing one strategy from their strategy set, players were allowed to choose a *probability distribution* over their strategy set.
- Assume each player's strategy set $S_i = \{s^1, s^2, ...s^I\}$ is finite. A *mixed strategy* m_i for Player i is a probability distribution over S_i:
- a set of probabilities $\{m_i(s^1), m_i(s^2), ..., m_i(s^I)\}$ such that
  - $m_i(s^j) \in [0, 1]$ for $j = 1, ..., I$
  - $\sum_{j=1}^{I} m_i(s^j) = 1$
- Let M_i denote the set of all possible mixed strategies for Player i.
- M_i is the $(I - 1)$-dimensional unit simplex.
- A *pure* strategy is a special case of m_i that assigns probability 1 to a single element of S_i, and zero probability to the other elements of S_i.

Mixed Strategies

- Let $M = \times_{i=1}^{N} M_i$ denote the set of joint mixed strategies.
- Then, an element of M, $m \in M$, is a joint mixed strategy.
- We assume that players rank $m \in M$ based on their *expected utility*:

$$u_i(m) = \sum_{s \in S} m_1(s_1)...m_N(s_N)u_i(s)$$

- where $s = (s_1, ..., s_N)$, and $m_1(s_1)...m_N(s_N)$ is the joint probability that s is played, based on m.

Mixed Strategy Nash Equilibrium

- A *mixed strategy Nash equilibrium* is an equilibrium where no player has an incentive to deviate by changing his mixed strategy.
- **Def 7.9**: Given a finite strategic form game $G = (S_i, u_i)_{i=1}^N$, a mixed joint strategy $\hat{m}$ is a mixed strategy Nash equilibrium if, for each player i:

$$u_i(\hat{m}) \geq u_i(m_i, \hat{m}_{-i}) \qquad \text{for all } m_i \in M_i$$

- Now, we will drop the word "mixed". When we say "strategy", we mean "mixed strategy" and when we say "Nash equilibrium", we mean "mixed strategy Nash equilibrium".
- If we want to specifically refer to a situation without randomization, we will say "pure strategy" and "pure strategy Nash equilibrium".

Example: Football Penalty Kick

	L	R
L	1,-1	-1,1
R	-1,1	1,-1

- Suppose Player 1's strategy is: play L with probability p and R with probability $1 - p$.
- Suppose Player 2's strategy is: play L with probability q and R with probability $1 - q$.
- Player 1's expected payoff to the pure strategy L is $q \cdot 1 + (1 - q) \cdot (-1) = 2q - 1$
- Player 1's expected payoff to the pure strategy R is $q \cdot (-1) + (1 - q) \cdot 1 = 1 - 2q$
- Player 1's expected payoff to the joint strategy ((p, 1-p), (q, 1-q)) is $p(2q - 1) + (1 - p)(1 - 2q) = (2p - 1)(2q - 1)$

Example: Football Penalty Kick

	L	R
L	1,-1	-1,1
R	-1,1	1,-1

- Player 1's expected payoff to the joint strategy $((p, 1-p), (q, 1-q))$ is $p(2q-1) + (1-p)(1-2q) = (2p-1)(2q-1)$
- Suppose $q < 0.5$. Then Player 1's unique best response is $p = 0$.
- Suppose $q > 0.5$. Then Player 1's unique best response is $p = 1$.
- Suppose $q = 0.5$. Then Player 1's set of best responses is $p \in [0, 1]$.

Example: Football Penalty Kick

- Player 2's expected payoff to the pure strategy L is $p \cdot (-1) + (1-p) \cdot 1 = 1 - 2p$
- Player 2's expected payoff to the pure strategy R is $p \cdot 1 + (1-p) \cdot (-1) = 2p - 1$
- Player 2's expected payoff to the joint strategy ((p, 1-p), (q, 1-q)) is $q(1-2p) + (1-q)(2p-1) = -(2p-1)(2q-1)$
- Suppose $p < 0.5$. Then Player 2's unique best response is $q = 1$.
- Suppose $p > 0.5$. Then Player 2's unique best response is $q = 0$.
- Suppose $p = 0.5$. Then Player 2's set of best responses is $q \in [0, 1]$.

Example: Football Penalty Kick

- The intersection of best responses is $p = 0.5, q = 0.5$.
- For each player, the expected payoff to each of their pure strategies is equal.
  - For Player 1, $1 - 2q = 2q - 1$
  - For Player 2, $1 - 2p = 2p - 1$
- Note that Player 1's expected payoffs are determined by Player 2, and vice versa.
- A game can have a (mixed) NE even if no pure NE exist.

Tests of Nash Equilibrium

- **Theorem 7.1**: The following statements are true if and only if $\hat{m}$ is a Nash equilibrium:
- **1:** For every player i:
  - $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ with positive probability in $\hat{m}_i$
  - $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ with zero probability in $\hat{m}_i$
- **2:** For every player i, $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$.
- For each player's strategy $\hat{m}_i$, the expected payoffs to elements of S_i played with positive probability must be equal.
- The expected payoffs to elements of S_i played with zero probability must be no greater than that for elements played with positive probability.

Some Properties of Expected Values

- Suppose we have a random variable X that can take two outcomes: x_1 with probability p, and x_2 with probability $1 - p$.
- The expected value of X, denoted $E(X)$, is $px_1 + (1 - p)x_2$.
- $E(X)$ always takes a value that is *between* x_1 and x_2.
- If $x_1 < x_2$, then $E(X)$ decreases linearly as p goes from 0 to 1. $E(X)$ is maximized at $p = 0$.
- If $x_1 > x_2$, then $E(X)$ increases linearly as p goes from 0 to 1. $E(X)$ is maximized at $p = 1$.
- If $x_1 = x_2$, then $E(X)$ is equal to $x_1 = x_2$ for all values of p. Any value of p maximizes $E(X)$.

Some Properties of Expected Values

- Suppose there are n possible outcomes, $x_1, x_2, ...x_n$, with probabilities $p_1, p_2, ...p_n$, where $\sum_i^n p_i = 1$, and each $p_i \geq 0$.
- $E(X) = p_1 x_1 + ... + p_n x_n$
- As before, the value of $E(X)$ will always be in between the smallest and largest values of x_i.
- If $p_i = 1$ and $p_j = 0$ for $i \neq j$, then $E(X) = x_i$.
- If there is a single largest x_i, then $E(X)$ is maximized when $p_i = 1$ and $p_j = 0$.

Some Properties of Expected Values

- Suppose there are multiple largest values: for example, suppose $x_1 = x_2 = ... = x_k > x_{k+1} > ... > x_n$.
- Then, $E(X)$ is maximized when all the probability is allocated to $x_1, ...x_k$ and zero probability is allocated to the other values:

$$p_1 + p_2 + ... + p_k = 1, p_{k+1} = p_{k+2} = ... = p_n = 0$$

- Conversely, suppose we know that $E(X)$ is maximized with respect to $p_1, ...p_n$, and that $p_1...p_k$ are nonzero, while the rest of p_i's are zero.
- Then, it must be that the x_i's corresponding to $p_1...p_k$ are equal, and greater than or equal to the other x_i's.

$$x_1 = x_2 = ... = x_k \geq x_j \qquad \text{for } j > k$$

Example 7.1: A Coordination Game

	B	S
B	2,1	0,0
S	0,0	1,2

- There are two pure strategy NE: (B, B) and (S, S).
- Let Player 1's probability of playing S be p, and Player 2's probability of playing B be q.
- Let's check that $p = q = 1/3$ is a NE.
- Player 1's expected payoff to pure strategies:
  - $q \cdot 2 + (1 - q) \cdot 0 = 2q = 2/3$
  - $q \cdot 0 + (1 - q) \cdot 1 = 1 - q = 2/3$

Example 7.1: A Coordination Game

	B	S
B	2,1	0,0
S	0,0	1,2

- Player 2's expected payoff to pure strategies:
  - $(1-p)\cdot 1 + p \cdot 0 = 1 - p = 2/3$
  - $(1-p)\cdot 0 + p \cdot 2 = 2p = 2/3$
- Therefore, the test is satisfied and this is a NE.
- Note that in this game, the expected payoff in the mixed NE is lower than in either of the pure strategy NE.
- In this game, the players would prefer to not behave unpredictably.

Example: Choosing Numbers

- Players 1 and 2 choose a positive integer from $1...K$.
- If the players choose the same number, Player 2 gets a payoff of -1 and Player 1 gets a payoff of 1.
- Otherwise, both players get a payoff of 0.
- First, show that one NE is if both players choose each integer with equal probability $1/K$.

Example: Choosing Numbers

- Denote Player i's expected payoff to pure strategy j as $E_i(j)$.
- Player 1's expected payoffs to his pure strategies $1, ..., K$ are:
- $E_1(1) = E_1(2) = ... = E_1(K) = 1/K$
- Player 2's expected payoffs to his pure strategies $1, ..., K$ are:
- $E_2(1) = E_2(2) = ... = E_2(K) = -1/K$
- All actions with positive probability have the same payoff, so the condition for a NE is satisfied.

Example: Choosing Numbers

- Show there is no other NE.
- Let Player 1's mixed strategy be $(p_1, ..., p_K)$.
- Let Player 2's mixed strategy be $(q_1, ..., q_K)$.
- Player 1's expected payoffs to his pure strategies $1, ..., K$ are:
- $E_1(1) = q_1, E_1(2) = q_2, ..., E_1(K) = q_K$
- Player 2's expected payoffs to his pure strategies $1, ..., K$ are:
- $E_2(1) = -p_1, E_2(2) = -p_2, ..., E_2(K) = -p_K$

Example: Choosing Numbers

- Player 1's expected payoff, given both players' mixed strategies, is:

$$p_1 q_1 + p_2 q_2 + \ldots + p_K q_K$$

- Player 2's expected payoff, given both players' mixed strategies, is the negative of Player 1's expected payoff:

$$-p_1 q_1 - p_2 q_2 - \ldots - p_K q_K$$

- Suppose that Player 1 does not place equal probability on each number: there exists a number i such that p_i that is strictly greater than the other p's.
- Player 2's expected payoff to playing i is $-p_i$, so Player 2 will put zero probability on $i : q_i = 0$.
- However, if $q_i = 0$, then Player 1's expected payoff to i is 0, and Player 1's best response is to put zero probability on i, a contradiction.

Existence of Nash Equilibrium

- **Theorem 7.2**: Every finite strategic form game has at least one Nash equilibrium.
- We will prove this by constructing a function with a *fixed point* that is the Nash equilibrium.
- A fixed point of a function $F()$ is a point x such that $x = F(x)$.

Existence of Nash Equilibrium

- Define $f : M \to M$ as follows: for each joint strategy $m \in M$, player i and his pure strategy j:

$$f_{ij}(m) = \frac{m_{ij} + \max(0, u_i(j, m_{-i}) - u_i(m))}{1 + \sum_{j'=1}^{n} \max(0, u_i(j', m_{-i}) - u_i(m))}$$

- Note that for every player i, $\sum_{j=1}^{n} f_{ij}(m) = 1$ and $f_{ij}(m) \geq 0$ for any j.
- Let $f_i(m) = (f_{i1}(m), \ldots, f_{in}(m))$. Therefore, $f_i(m)$ is a probability distribution over n elements, and $f_i(m) \in M_i$.
- Let $f(m) = (f_1(m), \ldots, f_N(m))$. Therefore, $f(m)$ is a joint probability distribution, and $f(m) \in M$.
- $f(m)$ is continuous in m.
- By Brouwer's fixed point theorem, f has a fixed point, $\hat{m}$.

Existence of Nash Equilibrium

- We can show that $f(\hat{m}) = \hat{m}$ is actually the Nash equilibrium.

$$\hat{m}_{ij} = \frac{\hat{m}_{ij} + \max(0, u_i(j, \hat{m}_{-i}) - u_i(\hat{m}))}{1 + \sum_{j'=1}^{n} \max(0, u_i(j', \hat{m}_{-i}) - u_i(\hat{m}))}$$

Incomplete Information

- So far, we have assumed that players are perfectly informed about the payoffs of all other players.
- However, in many real-life situations, there is uncertainty about the opponents' payoffs. For example:
  - When you buy an item from a seller, the seller knows the item's quality, but you do not.
  - When two people get into a competition, each person knows his own strength, but not the other person's.
- We will show how to specify this situation as a strategic form game by adding two additional elements.

Player Types

- First, for each player i, we introduce a finite set of "types", T_i, that the player might be.
- For example:
  - A firm might have two types, "low production cost" and "high production cost".
  - A competitor might have two types, "strong" and "weak".
- A player's payoffs for a given joint pure strategy now also depend on his type.
- Let $T = \times_{i=1}^{N} T_i$, the set of joint types.
- Player i's payoff function u_i maps $S \times T$ to a real number.

Beliefs as Probability Distributions over Types

- Second, each player has *beliefs* about what all other players' type may be.
- A belief is a probability distribution over the set of possible types.
- For example, suppose there are 2 players, and Player 2 has two possible types: "weak" and "strong".
- Player 1 has a belief about Player 2's type, $(p, 1-p)$, where p is Player 1's *subjective probability* that Player 2 is "weak".
  - If $p = 0$, then Player 1 is certain that Player 2 is "strong".
  - If $p = 0.5$, then Player 1 thinks that it is equally likely that Player 2 is "weak" or "strong".
  - If $p = 1$, then Player 1 is certain that Player 2 is "weak".
- If Player 1 also has more than one type, we need to specify (possibly) different beliefs about Player 2, for each of Player 1's type.

Example: A Coordination Game with Different Types

- Consider the coordination game we saw earlier, but now suppose that Player 2 can have two different types.
- Type-1 of Player 2 is the same as before, and prefers to choose the same strategy as Player 1.
- Type-2 of Player 2, on the other hand, prefers to choose a different strategy as Player 1.
- If Player 2 is Type-1, then the payoff matrix is:

	B	S
B	2,1	0,0
S	0,0	1,2

Example: A Coordination Game with Different Types

	B	S
B	2,0	0,2
S	0,1	1,0

- If Player 2 is Type-2, then the payoff matrix is as above.
- Let $p_1(t_1)$ denote Player 1's subjective probability that Player 2 is of Type-1. Then, $p_1(t_2) = 1 - p_1(t_1)$.
- We will assume that all players know their own types with certainty.
- Suppose $p_1(t_1) = p_1(t_2) = 0.5$.
- We will only look at pure strategies for what follows. However, Player 1 still uses expected payoffs, where the uncertainty now comes from the type of Player 2.

Example: A Coordination Game with Different Types

- We will treat the 2 types of Player 2 as separate players; a joint strategy for all players has 3 strategies, one each for Player 1, Player 2 Type-1, and Player 2 Type-2.
- We will denote a joint strategy as e.g. $(B, (B, S))$, where the first element is Player 1's strategy, and the second element is (Type 1's strategy, Type 2's strategy).
- Player 1's expected payoff to each joint strategy is:
  - $(B, (B, B))$: 0.5*2 + 0.5*2 = 2
  - $(B, (B, S))$: 0.5*2 + 0.5*0 = 1
  - $(B, (S, B))$: 0.5*0 + 0.5*2 = 1
  - $(B, (S, S))$: 0.5*0 + 0.5*0 = 0
  - $(S, (B, B))$: 0.5*0 + 0.5*0 = 0
  - $(S, (B, S))$: 0.5*2 + 0.5*1 = 0.5
  - $(S, (S, B))$: 0.5*1 + 0.5*0 = 0.5
  - $(S, (S, S))$: 0.5*1 + 0.5*1 = 1

Example: A Coordination Game with Different Types

- We will solve this as a three-player game. We can compute the best response functions and see if there is an intersection.
- The best responses of Player 1 are: $B_1(B,B)=B, B_1(B,S)=B, B_1(S,B)=B, B_1(S,S)=S$.
- The best responses of Player 2, both types, only depend on Player 1's strategy.
  - Type 1: $B_{21}(B)=B, B_{21}(S)=S$
  - Type 2: $B_{22}(B)=S, B_{22}(S)=B$
- $(B,(B,S))$ is a Nash equilibrium, since all players are playing best responses to the other players.
- In this equilibrium, Player 1 plays B, Player 2-Type 1 plays B, and Player 2-Type 2 plays S.

Game of Incomplete Information (Bayesian Game)

- **Def 7.10**: A game of incomplete information (also called a Bayesian game) is a tuple $G = (p_i, T_i, S_i, u_i)_{i=1}^{N}$, where for each player i, the set of types T_i is finite, $u_i : S \times T \to \mathbb{R}$, and for each $t_i \in T_i, p_i(\cdot|t_i)$ is a probability distribution on T_{-i}.
- Here, $p_i(t_{-i}|t_i)$ is the probability that the players aside from i have joint type t_{-i}, conditional on Player i's own type being t_i.
- This allows for the possibility that Player i's own type is not independent of the other players' types: if Player i knows he is a specific type, this may give him more information on the distribution of other players' types.

The Associated Strategic Form Game

- Let $G = (p_i, T_i, S_i, u_i)_{i=1}^N$ be a game of incomplete information.
- We will construct a strategic form game G^* in which each player type in G is a separate player.
- For each player $i \in \{1, ..., N\}$ and each Player i-type $t_i \in T_i$, let t_i be a player in G^* whose finite set of pure strategies is S_i.
- Let $s_i(t_i) \in S_i$ denote the a pure strategy chosen by player $t_i \in T_i$.
- The payoff to player t_i from the joint pure strategy $s^* = (s_1(t_1), ..., s_N(t_N))$ is:

$$v_{t_i}(s^*) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(s_1(t_1), ..., s_N(t_N), t_1, ..., t_N)$$

Bayesian-Nash Equilibrium

- A *Bayesian-Nash equilibrium* of a game of incomplete information is a Nash equilibrium of the associated strategic form game.
- By the existence of Nash equilibrium in finite strategic form games, every finite incomplete information game has at least one Bayesian-Nash equilibrium.

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- Homework #3 is due next week.
- The midterm will be on Nov. 4.
- Midterm will be open-book.
- Chapters 1, 2.1, 3, and 7 will be covered.