# Advanced Microeconomic Analysis, Lecture 5

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October 28, 2014

# Announcements

- Homework #3 is due today.
- The midterm will be next week.
- No class on Nov. 11, due to APEC. Class resumes on Nov. 18, with Prof. Jianye Yan.
- Midterm will be open-book.
- Chapters 1, 2.1, 3, and 7 will be covered.

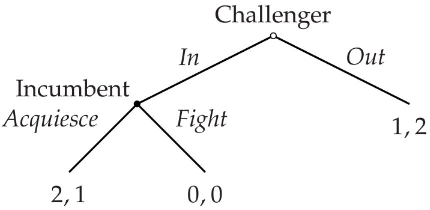
# Extensive Form Games

- So far, with strategic form games, we have been assuming that all players move simultaneously.
- This cannot capture a sequential situation, where one player can move after another.
- We will introduce a way of specifying a game with sequential moves.

# Example: An Entry Game

- Suppose we have a situation where there is an *incumbent* and a *challenger*.
- For example, an industry might have an established dominant firm.
- A challenger firm is deciding whether it wants to enter this industry and compete with the incumbent.
- If the challenger enters, the incumbent chooses whether to engage in intense (and possibly costly) competition, or to accept the challenger's entry.

# Game Tree




- We can represent this game with a tree diagram.
- The root node of the tree is the first move in the game (here, by the challenger).
- Each action at a node corresponds to a branch in the tree.
- Outcomes are leaf nodes (i.e. there are no more branches).
- The first number at each outcome is the payoff to the first player (the challenger).

# Histories (Nodes)

- A *history* is the sequence of actions played from the beginning, up to some point in the game.
  - In the tree, a history is a path from the root to some node in the tree. We will use "history" and "node" interchangeably.
  - In the entry game, all possible histories are: $\varnothing$ (i.e. at the beginning, no actions played yet), $(In), (Out), (In, Acquiesce), (In, Fight)$.
- A *terminal history* is a sequence of actions that specifies an outcome, which is what players have preferences over.
  - In the tree, a terminal history is a path from the root to an end node (a node with no branches).
  - We will use "terminal history" and "end node" interchangeably.
  - In the entry game, the terminal histories are: $(Out), (In, Acquiesce), (In, Fight)$.
- We will use $X$ to denote the set of all nodes, and $x \in X$ to denote a specific node.
- $x_0$ is the initial node, corresponding to the empty history $\varnothing$ at the beginning of the game.

# Actions

- Let $A$ denote the set of *all possible* actions that can be taken at any point in the game (can be infinite).
- Let $i(x)$ be a function that indicates whose turn it is to move at node $x$.
- Let $A(x)$ denote the set of actions that are available to player $i(x)$ at node $x$.
- Let $u_i(x)$ denote Player $i$'s von Neumann-Morgenstern utility function over the set of end nodes.

# Entry Game

- There are two players: *Incumbent* and *Challenger*.
- The challenger moves first, has two actions: *In* and *Out*.
- If the challenger chooses *In*, the incumbent chooses *Fight* or *Accept*.
- Payoff functions (challenger is $u_1$):

$$u_1(In, Acquiesce) = 2, u_1(Out) = 1, u_1(In, Fight) = 0$$

$$u_2(Out) = 2, u_2(In, Acquiesce) = 1, u_2(In, Fight) = 0$$

# Perfect Information vs. Imperfect Information

- Entry Game is an example of a game with *perfect information*: all players know the true history whenever it is their turn to move.
- A game with *imperfect information* is one where players may *not* know the true history.

# Buyer-Seller Game

- A buyer is deciding whether to purchase a used car from a seller.
- First, the seller decides whether to repair it or not. The buyer does not observe this action. Actions: $R, D$
- Then, the seller decides on a price to offer. Actions: $High, Low$
- Then, the buyer decides whether to accept the price or not. Actions: $Accept, Reject$

# Buyer-Seller Game

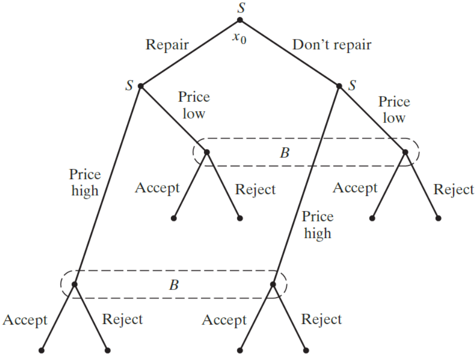



**Figure 7.10.** Buyer–seller game.

- The dashed lines show *information sets*.
- A player cannot distinguish between nodes in the same information set.

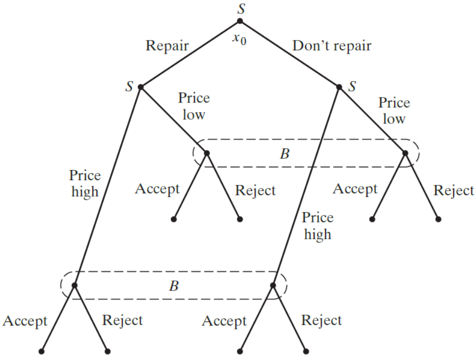



**Figure 7.10.** Buyer–seller game.

- The buyer does not know if the seller chose $R$ or $D$, so these nodes appear to be the same from the buyer's point of view.

# Information Sets

- $\mathcal{I}$ denotes a *partition* of the set of nodes, $X$.
- That is: $\mathcal{I}$ is a set of *disjoint* subsets of $X$, whose union is equal to $X$.
- If two nodes $x, x'$ are in the same partition, then it must be the same player's turn at $x, x'$, and the set of actions must be the same: $A(x) = A(x')$
- Each element of $\mathcal{I}$ is called an *information set*.

# Information Sets

- For example: in the Buyer-Seller game, the nodes are:

$$x_0, (R), (D),$$

$$(R, High), (R, Low), (D, High), (D, Low),$$

$$(R, High, Accept), (R, High, Reject), (R, Low, Accept), (R, Low, Reject),$$

$$(D, High, Accept), (D, High, Reject), (D, Low, Accept), (D, Low, Reject)$$

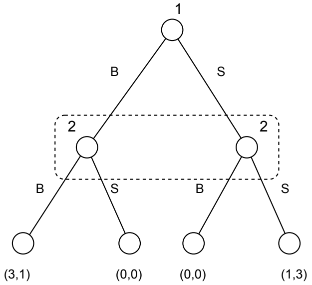
- $\{(R, Low), (D, Low)\}$ and $\{(R, High), (D, High)\}$ are the information sets with more than one element.
- All other information sets contain exactly one node.
- A game of perfect information is one where all information sets contain exactly one node.

# Simultaneous-Move Games

| | $B$ | $S$ |
|---|---|---|
| $B$ | 3,1 | 0,0 |
| $S$ | 0,0 | 1,3 |

- We can represent a simultaneous-move game as an extensive form game with imperfect information.

# Simultaneous-Move Games




- The coordination game above can be formulated as an extensive form game.
- The information set with more than one node is: $\{(B), (S)\}$.

# Extensive Form Game Strategy

- Player $i$'s pure strategy in an extensive form game specifies an action for every information set of Player $i$.
- Let $\mathcal{I}_i$ denote the set of Player $i$'s information sets.
- Let $s_i : \mathcal{I}_i \to A$ denote Player $i$'s pure strategy: a function that assigns an action to every information set of Player $i$.
- Let $S_i$ denote the set of pure strategies for Player $i$.

# Extensive Form Game Definition

- An extensive form game, denoted $\Gamma$, consists of:
  - A finite set of players, $N$.
  - A set of actions, $A$.
  - A set of nodes (histories), $X$.
  - A set of end nodes (terminal histories), $E$.
  - A function $i(x)$ that indicates whose turn it is to move at $x$.
  - A partition $\mathcal{I}$ of $X$ into information sets.
  - Payoff functions $u_i$ for each player that are von Neumann-Morgenstern.
- As we go along, we will see additional elements that can extend our definition of an extensive form game.

# Entry Game

- In the Entry Game, *Challenger* has one information set, $x_0$.
- 2 possible strategies: $In, Out$.
- *Incumbent* has one information set, $(In)$.
- 2 possible strategies: $Acquiesce, Fight$.

# Buyer-Seller Game

- *Seller* has three information sets: $x_0, (R), (D)$.
- A strategy must specify an action for 3 information sets.
- $2 \cdot 2 \cdot 2 = 8$ possible strategies.
- For example: the strategy $(R, High, Low)$, by convention, means:
  - At the first information set, $x_0$, choose $R$.
  - At the second information set, $(R)$, choose $High$.
  - At the third information set, $(D)$, choose $Low$.

# Buyer-Seller Game

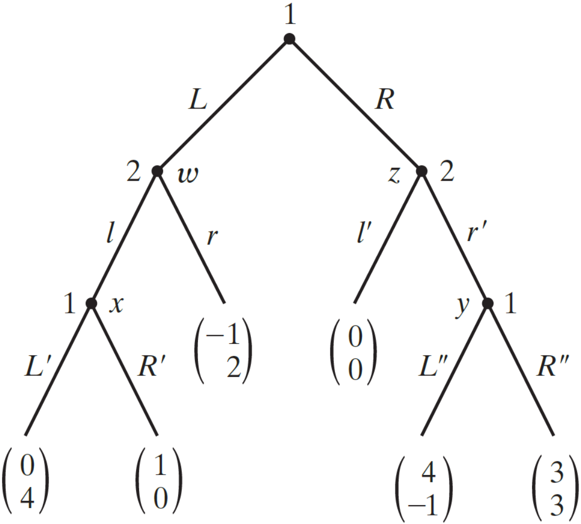
- *Buyer* has two information sets: $\{(R, High), (D, High)\}$ and $\{(R, Low), (D, Low)\}$.
- A strategy must specify an action for 2 information sets.
- $2 \cdot 2 = 4$ possible strategies.
- $(Accept, Accept)$, $(Accept, Reject)$, $(Reject, Accept)$, $(Reject, Reject)$.
- The first element of the pair specifies the action at the first information set.

# Strategic Form of an Extensive Game

- We can enumerate all possible strategies of all players.
- Suppose we list all strategies of Player $i$ as $S_i$. Then the tuple $(S_i, u_i)_{i \in N}$ defines a strategic form game.
- This is called the *strategic form* of $\Gamma$.
- We can then find Nash equilibrium of the strategic form.
- However, not all of these Nash equilibria will be considered "reasonable".

# Backward Induction Strategies

- Suppose we have a game with perfect information (all information sets have one node).
- Consider the Entry Game.
- In the last stage, the payoff-maximizing choice for *Incumbent* is *Acquiesce*.
- If *Challenger* takes this as given, then the payoff of choosing *In* becomes $(2, 1)$.
- Therefore, the payoff-maximizing choice for *Challenger* is *In*.
- The backwards induction strategy pair is $(In, Acquiesce)$.



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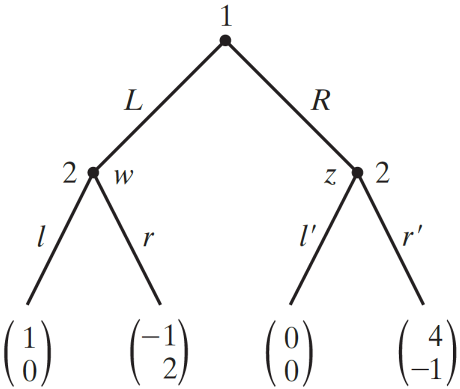
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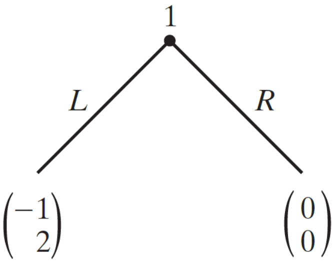
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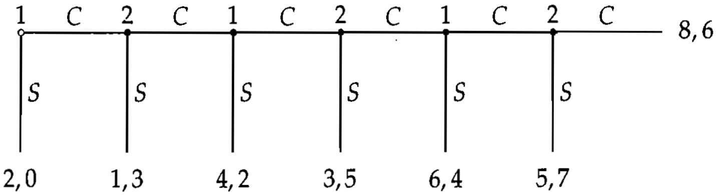
# Backward Induction Algorithm

- At the last stage, find the set of payoff-maximizing actions.
- At the second to last stage, assume that one of these actions will be played, and take the payoffs of that outcome as given.
- Repeat until actions have been determined for every decision node.

# Centipede Game



# Backward Induction and Nash Equilibrium

- A *finite* extensive form game is one where the number of actions and nodes are both finite.
- **Theorem 7.4**: If $s$ is a backward induction strategy for a perfect information, finite, extensive form game $\Gamma$, then $s$ is a Nash equilibrium of $\Gamma$.
- Since a payoff-maximizing action always exists for a finite game, a backward induction strategy (and therefore a NE) always exists for a finite game.
- However, a NE is not necessarily a backward induction strategy.

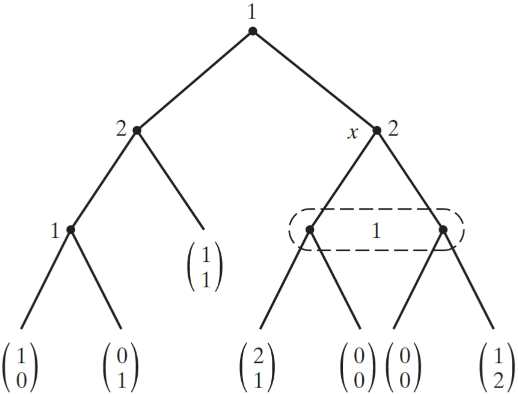
# NE of Entry Game

| | *Acquiesce* | *Fight* |
|---|---|---|
| *In* | 2,1 | 0,0 |
| *Out* | 1,2 | 1,2 |

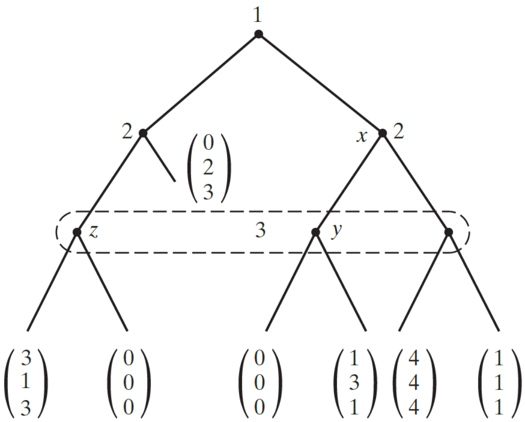
- There are two NE: $(In, Acquiesce)$ and $(Out, Fight)$.
- However, the second NE violates rationality at the last stage, since *Fight* is not the payoff-maximizing action.
- This is called an *incredible threat*: if *Challenger* believes *Incumbent* is rational, then he will not believe *Fight* will be played.
- Backward induction ensures that strategies are rational at all stages.

# Imperfect Information and Subgames

- We want to extend the backward induction technique to games with imperfect information.
- A node $x$ defines a *subgame* of an extensive form game if $\mathcal{I}(x) = \{x\}$, and whenever $y$ is a node following $x$ and $z$ is in the same information set as $y$, then $z$ also follows $x$.
- Or, in other words: the subtree starting at $x$ is a subgame if separating it from the rest of the tree does not split any information sets.
- A *proper* subgame is a subgame that is not the original game itself.




- This game has 3 proper subgames.

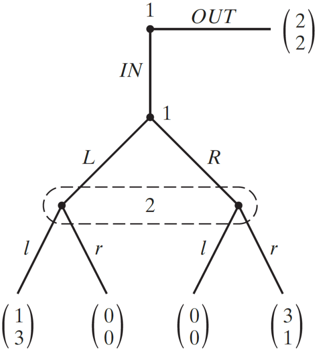



- This game has no proper subgames.

# Pure Strategy Subgame Perfect Equilibrium

- **Def 7.17**: A joint pure strategy $s$ is a pure strategy, *subgame perfect* equilibrium of the extensive form game $\Gamma$ if it induces a Nash equilibrium in every subgame of $\Gamma$.
- This generalizes backward induction to games with imperfect information.
- **Theorem 7.5**: For every finite extensive form game with perfect information, the set of backward induction strategies is the set of pure strategy subgame perfect equilibria.

# Coordination Game with Option

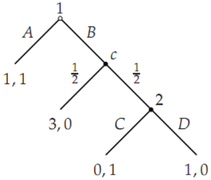



- The *NE* of the subgame starting after history $(In)$ are $(L, l)$ and $(R, r)$.
- For each of these strategies, take it as given and find the payoff-maximizing action at the first stage.
- The subgame perfect NE are: $((Out, L), l)$ and $((In, R), r)$.

# Allowing for exogenous uncertainty

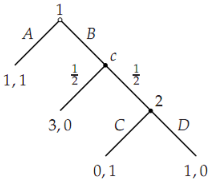
- So far, all the extensive games we've seen have been deterministic: no randomness in outcomes.
- We can allow randomness by introducing an additional player, called "Chance" or "Nature"
- Nature makes choices randomly, according to a known probability distribution.
- Players maximize expected payoffs.

# Allowing for exogenous uncertainty




- Consider this game with chance moves.
- Here, "$c$" is the Chance player, who chooses randomly between two branches, each with probability $1/2$.
- This is still a game with *perfect information*: at each player's move, he knows exactly what sequence of moves has occurred in the past.
- In the last subgame, Player 2 chooses $C$, so this subgame has a payoff of $(0, 1)$.

# Allowing for exogenous uncertainty




- In the subgame following $B$, Chance chooses each action with probability $\frac{1}{2}$.
- The expected payoff to Player 1 of this subgame is therefore $\frac{1}{2}3 + \frac{1}{2}0 = 1.5$.
- The expected payoff to Player 2 of this subgame is $\frac{1}{2}0 + \frac{1}{2}1 = 0.5$.
- At the beginning, Player 1 chooses $B$, since the expected payoff of $B$ is greater than the expected payoff of $A$.

# Mixed Strategies and Behavioral Strategies

- As before, we can allow randomization by players.
- There are two equivalent ways of specifying a randomized strategy for a player.
- First, specify a single probability distribution over the set of *all* pure strategies (which specify an action for every information set).
- As before, this is a *mixed strategy* for Player $i$.
- Second, specify a separate probability distribution at every information set of Player $i$.
- This is called a *behavioral strategy*.

# Beliefs as Probability Distributions over Histories

- If we allow randomization (by Nature, or by players) with imperfect information, then a player at an information needs to have *beliefs* about which node he is in.
- Players will maximize their expected payoffs based on the probabilities specified by their beliefs.
- We want beliefs to be correct at an equilibrium, that is, a player's beliefs about the probability of being at a node, should be consistent with the true probability distribution generated by Nature and by all players' strategies.
- Beliefs should be derived from strategies using Bayes' Rule whenever possible.

# Conditional Probability

- Let $U$ be the universe of outcomes, and let $A, B$ be *events* (i.e. subsets of $U$) with probabilities $P(A), P(B)$.
- If we know one event $A$ has occurred, does that affect the probability that another event $B$ has occurred?
- Suppose we are told that event $A$ has occurred. Then $P(A) > 0$, and every outcome outside of $A$ is no longer possible.
- Therefore, the universe is reduced to $A$. The only part of $B$ which can occur is $A \cap B$.
- Since total probability of the universe must equal 1, the probability of $A \cap B$ must be scaled by $\frac{1}{P(A)}$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, P(B \cap A) = P(A)P(B|A)$$

# Conditional Probability

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, P(B \cap A) = P(A)P(B|A)$$

- $B \cap A$ and $\neg B \cap A$ are disjoint, and their union is $A$. Therefore, $P(B \cap A) + P(\neg B \cap A) = P(A)$.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B \cap A)}{P(B \cap A) + P(\neg B \cap A)}$$

- $P(B|A)$ is the probability of $B$ conditional on $A$ occurring. Equivalently,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A \cap B) + P(\neg A \cap B)}, P(A \cap B) = P(B)P(A|B)$$

# Bayes' Theorem

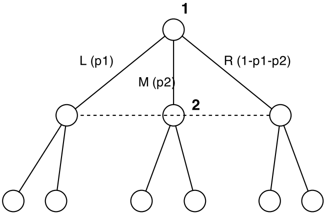
$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B \cap A)}{P(B \cap A) + P(\neg B \cap A)}, P(B \cap A) = P(A)P(B|A)$$

- Using the result $P(B \cap A) = P(B)P(A|B)$ and $P(\neg B \cap A) = P(\neg B)P(A|\neg B)$, we get

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\neg B)P(\neg B)}$$

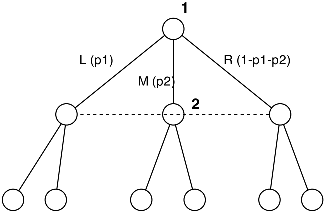
- This is known as Bayes' Theorem or Bayes' Rule. It is simply substituting in different formulas for $\frac{P(B \cap A)}{P(A)}$.
- Using this formula, we can calculate the probability of any event conditional on any other event, if we know the all of the other probabilities in the formula.

# Example 1



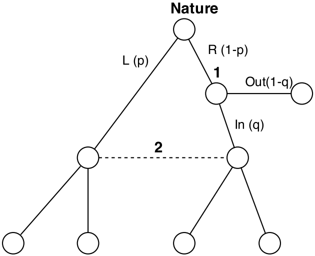

- Here, suppose Player 1 chooses $L, M, R$ randomly according to a known distribution: $P(L) = p_1, P(M) = p_2, P(R) = 1 - p_1 - p_2$.
- Player 1 may be a human player, or it may be Nature.
- Player 2 has one information set, consisting of the histories $\{L, M, R\}$.
- What is the correct belief at Player 2's information set?

# Example 1



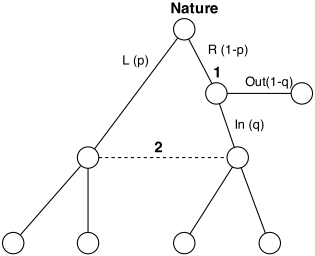

- The true probabilities generated by Player 1's behavioral strategy is simply the strategy itself. So, the only correct belief at Player 2's information set is $(p_1, p_2, 1 - p_1 - p_2)$.

# Example 2



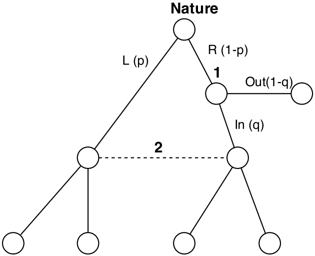

- In this game, Nature goes first and chooses $L$ with probability $p$, and $R$ with probability $1 - p$.
- If $R$ is chosen, then it is Player 1's move. He can choose *Out* and end the game, or choose *In*.
- Assume his behavioral strategy is to choose *In* with probability $q$, and *Out* with probability $1 - q$.

# Example 2




- Player 2 has one information set, consisting of the histories $\{L, (R, In)\}$.
- What is the correct belief at Player 2's information set?
- We will denote the event that Nature chose $L$ as $L$, and the event that Nature chose $R$ as $\neg L$.
- Denote the event that Player 1 chose $In$ as $In$, and the event that Player 1 chose $Out$ as $\neg In$.

# Example 2

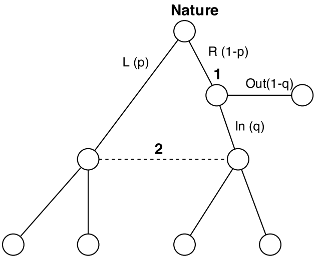



- Let $C$ denote the event that the game has reached Player 2's information set. This is true if one of the histories leading to Player 2's information set occurred.

$$P(C) = P(L) + P(R \cap In) = P(L) + P(R)P(In) = p + (1-p)q$$

- We want to find Player 2's correct beliefs about the relative probability of $L$ and $\neg L$, given that his information set has been reached.

# Example 2

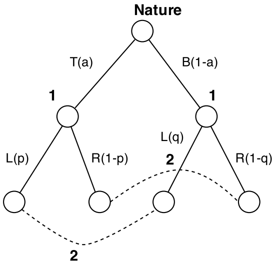



- This is equivalent to $P(L|C)$. Using Bayes' Theorem:

$$P(L|C) = \frac{P(L \cap C)}{P(C)} = \frac{P(L)}{P(C)} = \frac{p}{p + (1-p)q}$$

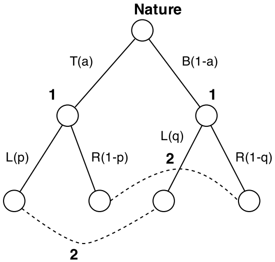
- Then, given these beliefs, Player 2 can then calculate the expected payoff to his actions at his information set.

# Example 3



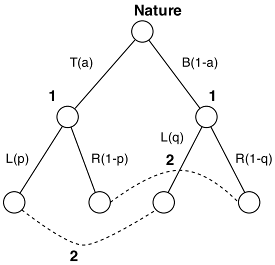

- In this game, Nature goes first and chooses $T$ with probability $a$, $B$ with probability $1 - a$.
- Player 1 observes this choice, and chooses $L$ or $R$.
- Player 2 does not observe Nature's choice, but only Player 2's choice.
- This setup is quite common in signaling games. Nature's choice determines the *type* of Player 1.

# Example 3



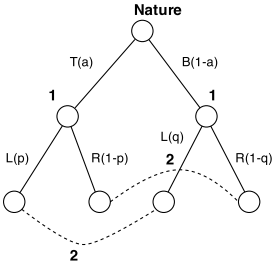

- Player 1 has two information sets: $\{L\}$ and $\{R\}$. At each of these information sets, he has the same two actions $L, R$.
- Player 2 has two information sets: the information set after $L$, $\{TL, BL\}$, and the information set after $R$, $\{TR, BR\}$.
- Let $T$ denote the event that Nature chose $T$, and $\neg T$ denotes the event that Nature chose $B$.
- Let $L|T$ denote the event that Player 1 chooses $L$ after observing $T$, and $\neg L|T$ denotes the event that Player 1 chooses $R$ after observing $T$.

# Example 3



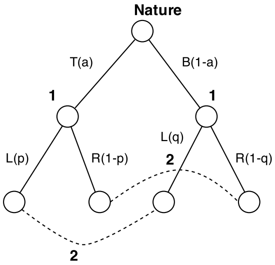

- Likewise, $L|\neg T$ is the event that Player 1 chooses $L$ after observing $B$, and $\neg L|\neg T$ is the event that Player 1 chooses $R$ after observing $B$.
- Suppose that Player 1's behavioral strategy specifies this probability distribution at each of his two information sets:
- $P(L|T) = p, P(\neg L|T) = 1 - p, P(L|\neg T) = q, P(\neg L|\neg T) = 1 - q$

# Example 3




- We want to find Player 2's correct beliefs at each of his two information sets.
- Consider the information set after $L$: this is $(T \cap L) \cup (\neg T \cap L)$.
- $P(L) = P(T)P(L|T) + P(\neg T)P(L|\neg T) = ap + (1-a)q$
- $P(T|L) = \frac{P(T \cap L)}{P(L)} = \frac{P(T)P(L|T)}{P(L)} = \frac{ap}{ap+(1-a)q}$

# Example 3




- Consider the information set after $R$: this is $(T \cap R) \cup (\neg T \cap R)$.
- $P(\neg L) = P(T)P(\neg L|T) + P(\neg T)P(\neg L|\neg T) = a(1-p) + (1-a)(1-q)$
- $P(T|\neg L) = \frac{P(T \cap \neg L)}{P(\neg L)} = \frac{P(T)P(\neg L|T)}{P(\neg L)} = \frac{a(1-p)}{a(1-p)+(1-a)(1-q)}$

# System of Beliefs

- Now, we know how to calculate beliefs at all information sets, which should allow us to calculate expected payoffs for each player's action.
- Let's state some definitions which will lead to our concept of equilibrium:
- A *system of beliefs* is a function that assigns to each information set, a probability distribution over the histories in that information set.
- A system of beliefs is simply a list of the currently moving player's belief, at every information set.

# Consistent Assessments

- An *assessment* is a pair $(p, b)$ of a system of beliefs $p$, and a joint behavioral strategy $b$.
- **Def 7.20** An assessment $(p, b)$ for a finite extensive form game $\Gamma$ is *consistent* if there is a sequence of completely mixed behavioral strategies $b^n \rightarrow b$ that converges to $b$, such that the associated sequence of Bayes' rule induced systems of beliefs, $p^n$, converges to $p$.
- A simpler concept of consistency is *weak consistency*: if
  - At information sets reached with positive probability by $b$, beliefs are derived from Bayes' Rule.
  - At information sets not reached with positive probability, beliefs may be anything.

# Sequential Rationality

- **Def 7.21** An assessment $(p, b)$ for a finite extensive form game $\Gamma$ is *sequentially rational* if for every player $i$, every information set $I$ belonging to player $i$, and every behavioral strategy $b'_i$ of player $i$,

$$v_i(p, b|I) \geq v_i(p, (b'_i, b_{-i})|I)$$

- In words: all players choose a payoff-maximizing action at all of their information sets, given the beliefs specified by $p$ at the information set.

# Sequential Equilibrium

- **Def 7.22** An assessment $(p, b)$ for a finite extensive form game $\Gamma$ a *sequential equilibrium* if it is both consistent and sequentially rational.
- An assessment $(p, b)$ for a finite extensive form game $\Gamma$ a *weak sequential equilibrium* if it is both weakly consistent and sequentially rational.

# Announcements

- Homework #3 is due today.
- The midterm will be next week.
- No class on Nov. 11, due to APEC. Class resumes on Nov. 18, with Prof. Jianye Yan.
- Midterm will be open-book.
- Chapters 1, 2.1, 3, and 7 will be covered.