Advanced Microeconomic Analysis

Solutions to Homework #2

0.1 1.41

Prove that Hicksian demands are homogeneous of degree 0 in prices. We use the relationship between Hicksian and Marshallian demands:

$$x_i^h(\boldsymbol{p}, u) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

where $e(\boldsymbol{p}, u)$ is the expenditure function. Then we use the fact that $e(\boldsymbol{p}, u)$ is homogeneous of degree 1 in $\boldsymbol{p}$, and Marshallian demand $x_i(\boldsymbol{p}, y)$ is homogeneous of degree 0 in $(\boldsymbol{p}, y)$:

$$x_i^h(t\boldsymbol{p}, u) = x_i(t\boldsymbol{p}, e(t\boldsymbol{p}, u)) = x_i(t\boldsymbol{p}, e(t\boldsymbol{p}, u)) = x_i(t\boldsymbol{p}, te(\boldsymbol{p}, u)) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

This is the same as the original value of $x_i^h(\boldsymbol{p}, u)$, so it is homogeneous of degree 0.

0.2 1.42

We use the Slutsky equation:

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_i} = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_i} - x_i(\boldsymbol{p}, u)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y}$$

The second term $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is always ≤ 0, and demand $x_i(\boldsymbol{p}, u)$ is always positive.

- Suppose x_i is a normal good. By definition, $\frac{\partial x_i}{\partial y} \geq 0$. Therefore, the sign of $-x_i(\boldsymbol{p}, u)\frac{\partial x_i(\boldsymbol{p},y)}{\partial y}$ is ≤ 0, so the sign of the left hand sign is ≤ 0. A decrease in own-price causes quantity demanded to increase.

  The converse of this statement is: if $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_j} \leq 0$, then x_i is a normal good. This depends on the relative magnitude of the two terms on the right-hand side; it may be possible for $\frac{\partial x_i}{\partial y}$ to be < 0 if the magnitude of $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is large. Therefore, the converse is not true.

- Suppose an own-price decrease causes a decrease in quantity demanded, i.e. $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_i} > 0$. Then the third term must be positive, so $\frac{\partial x_i(\boldsymbol{p},y)}{\partial y}$ must be negative, therefore x_i is an inferior good.

  The converse of this statement is: if x_i is inferior (therefore $\frac{\partial x_i(\boldsymbol{p},y)}{\partial y} < 0$), then $\frac{\partial x_i(\boldsymbol{p},y)}{\partial p_i} > 0$. This is not true if the magnitude of $\frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_j}$ is large enough. Therefore, the converse is not true.

0.3 1.54

$$u(x_1, ..., x_n) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \sum_{i=1}^{n} \alpha_i = 1$$

$$L(x_1, ..., x_n) = Ax_1^{\alpha_1}...x_n^{\alpha_n} - \lambda(p_1x_1 + ... + p_nx_n - y)$$

$$\frac{\partial L}{\partial x_i} = \frac{\alpha_i A x_1^{\alpha_1}...x_n^{\alpha_n}}{x_i} - \lambda p_i = 0 \qquad \text{for } i = 1...n$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + ... + p_nx_n - y = 0$$

$$\frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i} = \frac{p_i}{p_j} \Rightarrow x_j = \frac{p_i}{p_j}\frac{\alpha_j}{\alpha_i}x_i$$

Plugging into the budget equation:

$$p_1x_1 + p_2\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1 + ... + p_n\frac{p_1}{p_n}\frac{\alpha_n}{\alpha_1}x_1 = y$$

Marshallian demand:

$$x_i = \frac{y}{p_i\frac{\sum_j \alpha_j}{\alpha_i}} = \frac{\alpha_i y}{p_i}$$

Indirect utility:

$$u(\boldsymbol{x}^*) = A\left(\frac{\alpha_1 y}{p_1}\right)^{\alpha}_{1} ... \left(\frac{\alpha_n y}{p_n}\right)^{\alpha}_{n} = Ay\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} ... \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

Expenditure function: use the relationship $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$

$$Ay\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} = u \Rightarrow y = \frac{u}{A}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

Hicksian demand: differentiate $e(\boldsymbol{p}, u)$ with respect to p_i.

$$x_i^h(\boldsymbol{p}, u) = \frac{\alpha_i u}{Ap_i}\prod_{j=1}^{n}\left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}$$

0.4 1.56

- $v(p_1, p_2, p_3, y) = f(y)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. In order for this to be a legitimate indirect utility, it must satisfy the following conditions (all functions must be continuous):
  - Homogeneous of degree 0 in $(\boldsymbol{p}, y)$. Then $f(y)$ must be homogeneous of degree $-(\alpha_1 + \alpha_2 + \alpha_3)$
  - Strictly increasing in y. Then $f(y)$ must be strictly increasing.
  - Decreasing in $\boldsymbol{p}$. Then $\alpha_1, \alpha_2, \alpha_3$ must be ≤ 0.
  - Quasiconvex in $(\boldsymbol{p}, y)$. Then $(\alpha_1 + \alpha_2 + \alpha_3) \geq -1$ and $f(y)$ must be convex.

- $v(p_1, p_2, y) = w(p_1, p_2) + \frac{z(p_1,p_2)}{y}$.
  - Homogeneous of degree 0 in $(\boldsymbol{p}, y)$. Then $z(p_1, p_2)$ must be homogeneous of degree 1 and $w(p_1, p_2)$ must be homogeneous of degree 0.
  - Strictly increasing in y. This is always satisfied.
  - Decreasing in $\boldsymbol{p}$. $w(p_1, p_2)$ and $z(p_1, p_2)$ must be decreasing.
  - Quasiconvex in $(\boldsymbol{p}, y)$. $w(p_1, p_2)$ and $z(p_1, p_2)$ must be quasiconvex.

0.5 2.3

Given $v(\boldsymbol{p}, y) = yp_1^\alpha p_2^\beta$, $\alpha, \beta < 0$. The direct utility is:

$$
\begin{aligned}
u(\boldsymbol{x}) &= \min_{\boldsymbol{p}} v(\boldsymbol{p}, 1) \qquad \text{s.t. } \boldsymbol{p} \cdot \boldsymbol{x} = 1 \\
&= \min_{\boldsymbol{p}} p_1^\alpha p_2^\beta \qquad \text{s.t. } p_1x_1 + p_2x_2 = 1 \\
L(p_1, p_2, \lambda) &= p_1^\alpha p_2^\beta - \lambda(p_1x_1 + p_2x_2 - 1) \\
\frac{\partial L}{\partial p_1} &= \alpha p_1^{\alpha-1} p_2^\beta - \lambda x_1 = 0 \\
\frac{\partial L}{\partial p_2} &= \beta p_1^\alpha p_2^{\beta-1} - \lambda x_2 = 0 \\
\frac{\partial L}{\partial \lambda} &= p_1x_1 + p_2x_2 - 1 = 0 \\
\frac{\alpha}{\beta} &= \frac{p_1x_1}{p_2x_2} \Rightarrow p_2 = \frac{\beta}{\alpha}\frac{x_1}{x_2}, p_1 = \frac{\alpha}{\beta}\frac{x_2}{x_1}p_2 \\
p_1x_1 + p_1\frac{x_1\beta}{x_2\alpha}x_2 &= 1 \Rightarrow p_1 = \frac{1}{x_1(1+\frac{\beta}{\alpha})}, p_2 = \frac{1}{x_2(1+\frac{\alpha}{\beta})} \\
u(x_1, x_2) &= \left(\frac{1}{x_1(1+\frac{\beta}{\alpha})}\right)^\alpha \left(\frac{1}{x_2(1+\frac{\alpha}{\beta})}\right)^\beta
\end{aligned}
$$

0.6 2.5

Suppose $e(\boldsymbol{p}, u) = up_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$.

We use the relation $e(\boldsymbol{p}, v(\boldsymbol{p}, u)) = y$:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, u)) = v(\boldsymbol{p}, u)p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} = y \Rightarrow v(\boldsymbol{p}, u) = yp_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}$$

Roy's identity states that $x_i(\boldsymbol{p}, y) = \frac{-\partial v/\partial p_i}{\partial v/\partial y}$. Using our derived $v(\boldsymbol{p}, y)$:

$$\frac{-\partial v/\partial p_i}{\partial v/\partial y} = \frac{-\alpha^i yp_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{p_i p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_i y}{p_i}$$

which is the same as the demands that generated $e(\boldsymbol{p}, u)$.

To construct the utility function $u(\boldsymbol{x})$, we use:

$$u(\boldsymbol{x}) = \max\{u \geq 0 | \boldsymbol{p} \cdot \boldsymbol{x} \geq e(\boldsymbol{p}, u) \qquad \forall \boldsymbol{p} >> 0\}$$
$$= \max u \qquad \text{s.t. } \boldsymbol{p} \cdot \boldsymbol{x} \geq u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$$

u is maximized when the constraint holds with equality. Setting $y = \boldsymbol{p} \cdot \boldsymbol{x}$:

$$u \frac{y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}} = \frac{y}{\left(\frac{\alpha_1 y}{x_1}\right)^{\alpha_1} \left(\frac{\alpha_2 y}{x_2}\right)^{\alpha_2} \left(\frac{\alpha_3 y}{x_3}\right)^{\alpha_3}} = \frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3}}$$

This utility is of the Cobb-Douglas form. From problem 1.54, we know that Marshallian demands are of the form $x_i = \frac{\alpha_1 y}{p_i}$, which matches the original demand functions in the problem.

0.7 3.7

Suppose the N inputs are partitioned into groups $N_1, ... N_S$. The condition for weak separability is:

$$\frac{\partial(f_i/f_j)}{\partial x_k} = 0 \qquad \text{for all } i, j \in N_s, k \notin N_s$$

Let's check that is holds for the given function. Suppose input i is in group s, and let g_j denote the partial derivative of g with respect to its j-th argument:

$$f_i = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = g_s(f^1(\boldsymbol{x}^{(1)}, ..., f^S(\boldsymbol{x}^{(S)}))) \frac{\partial f^s(\boldsymbol{x}^{(s)})}{\partial x_i}$$

Then if i, j are in group s:

$$\frac{f_i}{f_j} = \frac{\frac{\partial f^s(\boldsymbol{x}^{(s)})}{\partial x_i}}{\frac{\partial f^s(\boldsymbol{x}^{(s)})}{\partial x_j}}$$

Since x_k for $k \notin N_s$ is not an argument of f^s, then $\frac{\partial(f_i/f_j)}{\partial x_k} = 0$ and weak separability is verified.

The condition for strong separability is:

$$\frac{\partial(f_i/f_j)}{\partial x_k} = 0 \qquad \text{for all } i \in N_s, j \in N_t, k \notin N_s \cup N_t$$

For the second function, suppose that input $i \in N_s$.

$$f_i = g'(f^1(\boldsymbol{x}^{(1)} + ... + f^S(\boldsymbol{x}^{(S)}))) \frac{\partial f^s(\boldsymbol{x}^{(s)})}{\partial x_i}$$

Suppose that input $j \in N_t, t \neq s$. Then

$$f_j = g'(f^1(\boldsymbol{x}^{(1)} + ... + f^S(\boldsymbol{x}^{(S)}))) \frac{\partial f^t(\boldsymbol{x}^{(t)})}{\partial x_j}$$

$$\frac{f_i}{f_j} = \frac{\frac{\partial f^s(\boldsymbol{x}^{(s)})}{\partial x_i}}{\frac{\partial f^t(\boldsymbol{x}^{(t)})}{\partial x_j}}$$

The only inputs that appear as arguments to this function are the inputs in N_s and N_t, therefore the derivative with respect to any other input is 0. This verifies the strong separability property.

0.8 3.17

Suppose $f(x_1, ...x_n) = (\sum_{i=1}^n \alpha_i x_i^\rho)^{\frac{1}{\rho}}, \sum_{i=1}^n \alpha_i = 1, 0 \neq \rho < 1$.

$$f_i = \left(\sum_{i=1}^n \alpha_i x_i^\rho\right)^{\frac{1}{\rho}-1} \alpha_i x_i^{\rho-1}$$

$$MRTS_{ij} = \frac{f_i}{f_j} = \frac{\alpha_i}{\alpha_j}\left(\frac{x_i}{x_j}\right)^{\rho-1}$$

Suppose $rho \to 0$. Then $MRTS_{ij} \to \frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i}$, which is the same as the $MRTS$ of the Cobb-Douglas form $y = \prod_{i=1}^n x_i^{\alpha_i}$.

Take the log of the MRTS:

$$\log(MRTS_{ij}) = \log(\alpha_i) - \log(\alpha_j) + (\rho - 1)(\log(x_i) - log(x_j))$$

As $\rho \to -\infty$, the sign of $(\log(x_i) - log(x_j))$ determines whether this goes to $+\infty$ or $-\infty$. If $\log(x_i) > log(x_j)$ (therefore $x_i > x_j$), the value goes to $-\infty$. If $\log(x_i) < log(x_j)$ (therefore $x_i < x_j$), the value goes to ∞. If $\log(x_i) = \log(x_j)$, the value is a finite constant. Therefore, the shape of the indifference curves must be flat whenever $x_i \neq x_j$. These are the indifference curves of the Leontief function $y = \min(x_1, ..., x_n)$.

0.9 3.28

(a) For inputs x_2, x_3, demand is increasing in the price of x_1. This is consistent with cost minimization, for example, with a Cobb-Douglas production function.

(b) This is consistent with a production function where x_1, x_2 are substitutes and x_1, x_3 are complements: for example, $y = x_2 + x_1^\alpha x_3^{1-\alpha}$.

(c) As output increases, demand for all output decreaes. This contradicts our assumption that production is strictly increasing, so this is not consistent with cost minimization.

(d) This is consistent with the Leontief production function; after a certain point, increasing one input does not affect the output.

(e) This is consistent with a production function that is separable such that the ratio of x_1/x_2 does not depend on x_3.

0.10 3.32

Let the cost function be $c(q)$. Then average cost is: $AC(q) = c(q)/q$ and marginal cost is: $MC(q) = c'(q)$. Suppose $AC(q)$ is declining: then $AC'(q) = c'(q)/q - c(q)/q^2 < 0 \Rightarrow c'(q)/q < c(q)/q^2 \Rightarrow c'(q) < c(q)/q$. When $AC(q)$ is constant, $c'(q) = c(q)/q$, and when $AC(q)$ is increasing, $c'(q) > c(q)/q$.