Advanced Microeconomic Analysis

Solutions to Homework #3

0.1 3.42

By Theorem 3.5, we can recover the production function by solving the maximization problem

$$f(x_1, x_2) = \max_{w_1, w_2} y \qquad \text{s.t. } w_1 x_1 + w_2 x_2 = c(w_1, w_2, y)$$

Suppose $c(w_1, w_2, y) = yAw_1^{\alpha} w_2^{1-\alpha}$. Then the problem is

$$f(x_1, x_2) = \max_{w_1, w_2} y \qquad \text{s.t. } w_1 x_1 + w_2 x_2 = yAw_1^{\alpha} w_2^{1-\alpha}$$

$$= \max_{w_1, w_2} \frac{w_1 x_1 + w_2 x_2}{Aw_1^{\alpha} w_2^{1-\alpha}}$$

The first-order conditions are

$$\frac{\partial}{\partial w_1} = \frac{x_1}{Aw_1^{\alpha} w_2^{1-\alpha}} - \frac{\alpha(w_1 x_1 + w_2 x_2)}{w_1 A w_1^{\alpha} w_2^{1-\alpha}} = 0$$

$$\frac{\partial}{\partial w_1} = \frac{x_2}{Aw_1^{\alpha} w_2^{1-\alpha}} - \frac{(1-\alpha)(w_1 x_1 + w_2 x_2)}{w_2 A w_1^{\alpha} w_2^{1-\alpha}} = 0$$

Combining these two, we get

$$\frac{w_1 x_1}{\alpha} = \frac{w_2 x_2}{1-\alpha}$$

which is the same first-order conditions as the Cobb-Douglas production function $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Suppose $c(w_1, w_2, y) = yA(w_1^r + w_2^r)^{\frac{1}{r}}$. Then the problem is

$$f(x_1, x_2) = \max_{w_1, w_2} y \qquad \text{s.t. } w_1 x_1 + w_2 x_2 = yA(w_1^r + w_2^r)^{\frac{1}{r}}$$

$$= \max_{w_1, w_2} \frac{w_1 x_1 + w_2 x_2}{A(w_1^r + w_2^r)^{\frac{1}{r}}}$$

The first-order conditions are:

$$\frac{\partial}{\partial w_1} = \frac{x_1}{A(w_1^r + w_2^r)^{\frac{1}{r}}} - \frac{w_1 x_1 + w_2 x_2}{w_1^{r-1} A(w_1^r + w_2^r)^{\frac{1}{r}+1}} = 0$$

$$\frac{\partial}{\partial w_2} = \frac{x_2}{A(w_1^r + w_2^r)^{\frac{1}{r}}} - \frac{w_1 x_1 + w_2 x_2}{w_2^{r-1} A(w_1^r + w_2^r)^{\frac{1}{r}+1}} = 0$$

Combining these two, we get

$$\frac{x_2}{w_2^{r-1}} = \frac{x_1}{w_1^{r-1}}$$

which is the same first-order conditions as the CES production function.

0.2 3.55

The problem is to minimize total cost:

$$\min_{k,F_1,F_2} w_k k + w_F(F_1 + F_2) \qquad \text{s.t. } (kF_1)^{\frac{1}{2}} = 4, (kF_2)^{\frac{1}{2}} = 3$$

Rearranging the constraints, we get $F_1 = 16/k, F_2 = 9/k$. Plugging this into the objective function, the problem becomes

$$\min_k w_k k + w_F \frac{25}{k}$$

with first-order condition

$$\frac{\partial}{\partial k} = w_k - w_F \frac{25}{k^2} = 0$$

The solution is $k = 5\left(\frac{w_F}{w_k}\right)^{\frac{1}{2}}$.

0.3 7.3

	L	M	R
U	2,1	1,1	0,0
C	1,2	3,1	2,1
D	2,-2	1,-1	-1, -1

(a) In this game, U weakly dominates D and M weakly dominates R. Suppose we eliminate D, then M, then R. We are left with the 2×1 game consisting of $\{U, C\}$ and L.

Suppose instead, we eliminate R. There are no further weakly dominated strategies, so what remains is the 3×2 game consisting of $\{U, C, D\}$ and $\{L, M\}$.

(b) First, we will prove 7.2: For a finite game, iterative elimination of strictly dominated strategies (IESDS) terminates in a finite number of rounds. Since there are a finite number of strategies, and every round of elimination must remove at least one strategy, and the process of elimination must terminate if each player has only 1 strategy left, then IESDS terminates in a finite number of rounds.

Now, suppose we have a finite game with at least one strictly dominated strategy. Let A be a strictly dominated strategy. Then, any subset of A must also be strictly dominated; that is, no matter what other strategies are eliminated, A remains strictly dominated. Since IESDS terminates in a finite number of rounds, then IESDS will eventually eliminate A in a finite number of rounds. Therefore, A will always be eliminated by IESDS, no matter what the order of elimination is.

We have established that all strategies that are strictly dominated at the beginning of IESDS will eventually be eliminated, and that once a strategy becomes strictly dominated, it will remain strictly dominated for the rest of IESDS until it is eliminated. Therefore, any strategy that becomes strictly dominated in the process of IESDS will eventually be eliminated.

Finally, let A^0 be the set of strategies that are strictly dominated at the beginning of IESDS, A^1 be the set of remaining strategies that are strictly dominated after all strategies in A^0 have been eliminated, A^2 be the set of remaining strategies that are strictly dominated after all strategies in A^0 and A^1 have been eliminated, etc. Since IESDS terminates in a finite number of rounds, there is some $k > 0$ for which $A^k, A^{k+1}, ...$ are empty. IESDS will eventually eliminate all strategies in $A^0, A^1, ..., A^{k-1}$ no matter what the order of elimination is.

0.4 7.6

Suppose we have a sequence of eliminated, strictly dominated strategies $a_1, a_2, ...a_n$. Since a strictly dominated strategy is also a weakly dominated strategy, this is also a sequence of eliminations of weakly dominated strategies. Suppose we have an strategy A such that there exists a sequence of eliminations of strictly dominated strategies $a_1, ...a_n$, that eliminates A. Then, there exists a sequence of eliminations of weakly dominated strategies that eliminates A (the same sequence). Taking the contrapositive, if A is such that there *does not* exist a sequence of eliminations of weakly dominated strategies that eliminates A, then there does not exist a sequence of eliminations of strictly dominated strategies that eliminates A.

0.5 7.10

(a) There are two pure NE, (U, L) and (D, R). (U, L) is one where neither player plays a weakly dominated strategy.

Let's find the set of mixed strategy NE. Let p be Player 1's probability of playing U, and q be Player 2's probability of playing R. The expected payoffs to pure strategies of each player are:

$$E_1(U) = q, E_1(D) = 0$$

$$E_2(L) = p, E_2(R) = 0$$

If $E_1(U) < E_1(D)$, then Player 1's best response is $p = 0$. If $E_1(U) = E_1(D)$, all $p \in [0, 1]$ is a best response. If $E_1(U) > E_1(D)$, Player 1's best response is $p = 1$.

$E_1(U) = E_1(D)$ when $q = 0$ (therefore all $p \in [0, 1]$ is a best response of Player 1); when $q > 0$, Player 1's best response is $p = 1$. Likewise, $E_2(L) = E_2(R)$ when $p = 0$ (therefore all $q \in [0, 1]$ is a best response of Player 2); when $p > 0$, Player 2's best response is $q = 1$. The only intersections are $p = q = 0$ and $p = q = 1$, which are the two pure NE we have already found.

(b) There are three pure NE, (U, L), (U, R), and (D, L). Only (U, L) has neither player playing a weakly dominated strategy. Let p be Player 1's probability of playing U, and q be Player 2's probability of playing R. The expected payoffs to pure strategies of each player are:

$$E_1(U) = q, E_1(D) = 2q - 1$$

$$E_2(L) = p, E_2(R) = 2p - 1$$

$E_1(U) = E_1(D)$ when $q = 1$ (therefore all $p \in [0,1]$ is a best response of Player 1); when $q < 1$, Player 1's best response is $p = 1$. Likewise, $E_2(L) = E_2(R)$ when $p = 1$ (therefore all $q \in [0,1]$ is a best response of Player 2); when $p < 1$, Player 2's best response is $q = 1$. The set of intersections is : $p = 1, q \in [0,1]$ and $p \in [0,1], q = 1$. Any completely mixed strategy (i.e. not a pure strategy) is weakly dominated by the pure strategy U (for Player 1) or L (for Player 2). Therefore, the only NE where no player plays a weakly dominated strategy is (U, L), which we have already found.

(c) M is strictly dominated by L and can be eliminated. The pure NE are (U, l), (C, m), and (D, L).

0.6 7.14

- If all players choose P: (P, P, P), the payoff vector is $(-1, -1, -1)$.
- If one player chooses D, e.g. (D, P, P), the payoff vector is $(0, -1, -1)$.
- If two players choose D, e.g. (D, D, P), the payoff vector is $(-3, -3, -4)$.
- If three players choose D, (D, D, D), the payoff vector is $(-3, -3, -3)$.

First, we find the pure strategy NE.

- (P, P, P) is not a NE, since any player can increase his payoff from -1 to 0 by switching to D.
- When 1 player chooses D, e.g. (D, P, P), it is a NE, since the D player will decrease his payoff from 0 to -1 by switching, and the P player will decrease his payoff from -1 to -3 by switching.
- When 2 players choose D, e.g. (D, D, P), it is not a NE, since a D player can increase his payoff from -3 to -1 by switching.
- (D, D, D) is a NE, since any player will decrease his payoff from -3 to -4 by switching.

Now, we find the mixed NE. We take the position of Player 1. Let p, q, r be the probability that Players 1, 2, 3 respectively, play P. The probability that the other two players play (P, P) is qr. The probability that the other players play (P, D) or (D, P) is $(q(1-r) + (1-q)r)$. The probability that the other players play (D, D) is $(1-q)(1-r)$. The expected payoff to pure strategies is:

$$E_1(P) = qr(-1) + (q(1-r) + (1-q)r)(-1) + (1-q)(1-r)(-4)$$

$$E_1(D) = qr(0) + (q(1-r) + (1-q)r)(-3) + (1-q)(1-r)(-3)$$

Equating these two gives the condition $q = \frac{3r-1}{6r-3}$. In order for p, q, r to be a NE, this relationship must hold between any pair of p, q, r.

Suppose all three players play the same mixed strategy; then $p = q = r$. Then $r = \frac{3r-1}{6r-3}$, giving $p = q = r = 0.2113$.

Suppose one player (say, Player 2) plays pure P. Then $1 = \frac{3r-1}{6r-3}$ and $q = \frac{3-1}{6-3}$, which is satisfied when one of p, q, r is 1 and the other two are $\frac{2}{3}$.

0.7 7.17

(a) Neither player has a strictly dominant strategy.

(b) *Spy* is weakly dominant.

(c) The only pure strategy NE is $Destroy, Spy$ which uses a weakly dominant strategy. The other NE is where Player 1 plays *Destroy*, and Player 2 plays a mixed strategy of $(0.5, 0.5)$. Since a mixed strategy is never weakly dominant, this is the NE where no player plays a weakly dominant strategy.

0.8 7.18

We will denote *Spy* as S and *Don't Spy* as D.

(a) For the "aggressive" type, $Keep$ is strictly dominant, so it will always be played.

(b) Let p be the probability that the "non-aggressive" type of Player 1 plays $Keep$. Player 2's expected payoff to pure strategies are:

$$E_2(S) = \epsilon(-9) + (1-\epsilon)(p + 2(1-p))$$

$$E_2(D) = \epsilon(-1) + (1-\epsilon)(-p + 2(1-p))$$

$E_2(S) = E_2(D)$ when $p = \frac{4\epsilon}{1-\epsilon}$.

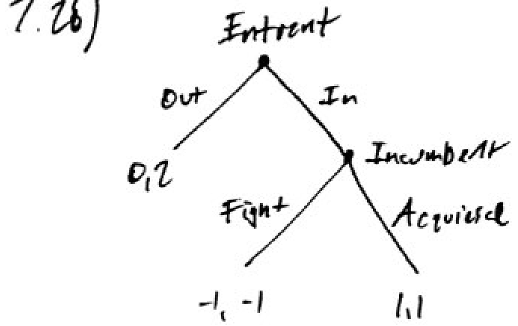
Let us call the "non-agressive" type as Player 3. Let q be the probability that Player 2 plays S. Player 3's expected payoffs to pure strategies are:

$$E_3(Keep) = -q + (1-q) = 1 - 2q$$

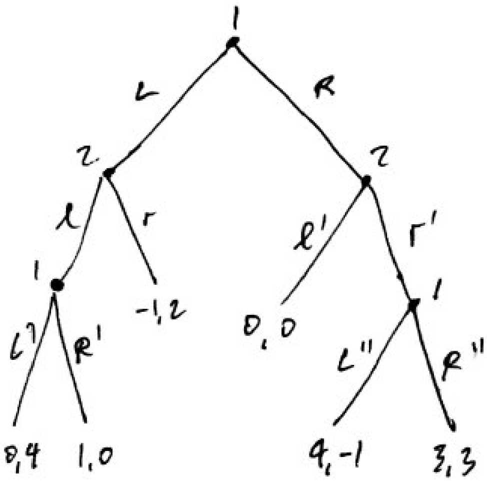
$$E_3(Destroy) = 0$$

$E_3(Keep) = E_3(Destroy)$ when $q = \frac{1}{2}$. Therefore, a Bayesian-Nash equilibrium is when $p = \frac{4\epsilon}{1-\epsilon}, q = \frac{1}{2}$.

7.26)



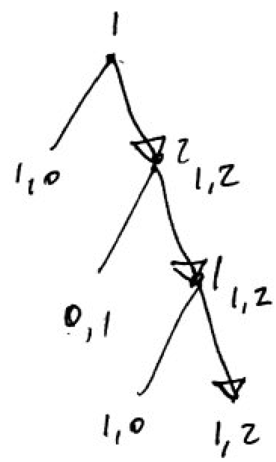
	Fight	Acquiesce
Out	0,2	0,2
In	-1,-1	1,1



	ll'	lr'	rl'	rr'
L L' L''	0,4	0,4	-1,2	-1,2
L L' R''	0,4	0,4	-1,2	-1,2
L R' L''	1,0	1,0	-1,2	-1,2
L R' R''	1,0	1,0	-1,2	-1,2
R L' L''	0,0	4,-1	0,0	4,-1
R L' R''	0,0	3,3	0,0	3,3
R R' L''	0,0	4,-1	0,0	4,-1
R R' R''	0,0	3,3	0,0	3,3

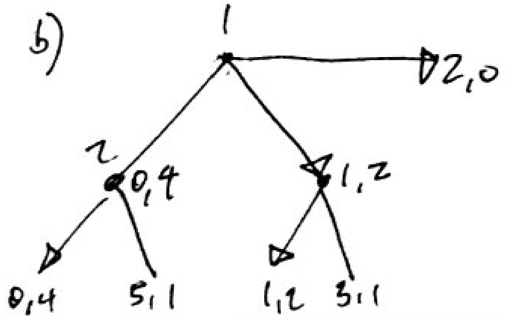
7.28)

a)



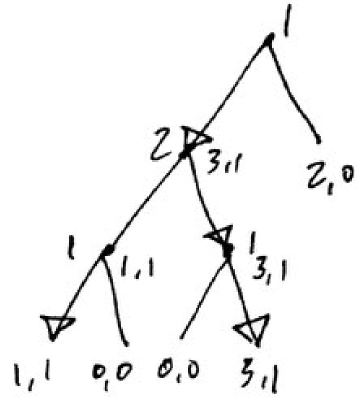
This has multiple backwards induction strategies.

b)



This has only one backward induction strategy.

c)



This has multiple strategies.

b) In the last stage, there is a unique payoff-maximizing choice, by the assumption that no player is indifferent between any pair of end nodes.
Therefore, the previous stage also has a unique choice.
By induction, this holds for every stage in the game.

c)