Topics in Bank Management

Solutions to Homework #1

0.1 8.6

(a) Let $u(w)$ denote the consumer's utility function, which is increasing and strictly concave. Expected utility is $\pi u(w_1)+(1-\pi)u(w_2)$. From the definition of strict concavity, we have

$$\pi u(w_1) + (1-\pi)u(w_2) < u(\pi w_1 + (1-\pi)w_2)$$

Let w' denote the certainty equivalent of this lottery, i.e. w' satisfies $u(w') = \pi u(w_1) + (1-\pi)u(w_2)$. Then $w' < \pi w_1 + (1-\pi)w_2$. Choose any wealth level z strictly in between w' and $\pi w_1 + (1-\pi)w_2$. Suppose an insurance company offers to pay the consumer z in both states; in exchange, the insurance company will receive the consumer's uncertain income (w_1 in state 1, w_2 in state 2). Then the consumer is strictly better off, since z provides a higher utility, and the insurance company makes a positive expected profit, since $z < \pi w_1 + (1-\pi)w_2$.

(b) Suppose there are a group of consumers $i = 1, ..., I$ that are insuring each other. Before insuring each other, consumer i's wealth is w_1^i in state 1 and w_2^i in state 2. After insuring each other, consumer i receives constant wealth z_i across both states. Then, it must be that $\sum_i z_i = \sum_i w_1^i = \sum_i w_2^i$. Suppose an insurance company offers to pay each consumer in this group the same constant amount z_i, in exchange for receiving their uncertain incomes $\sum_i w_1^i$ and $\sum_i w_2^i$. Then the consumers have the same allocation of wealth, and the insurance company makes zero expected profit.

0.2 8.16

We can model this situation as an extensive form game:

1. The owner offers a contract;
2. The worker chooses to accept or reject. If the worker rejects, he receives a utility corresponding to wage=0 and $e = 0$, which gives a utility of 0
3. The worker chooses $e = 0$ or $e = 1$.

This can be solved via backwards induction.

(a) Suppose e is observable by the owner, so the contract specifies the wage conditional on e: w^0, w^1 for $e = 0, e = 1$ respectively. In the last stage, the worker's utility is:

- If $e = 0$, $\sqrt{w^0} - 0 = \sqrt{w^0}$
- If $e = 1$, $\sqrt{w^1} - 1$

The worker will choose $e = 1$ if $\sqrt{w^1} - 1 \geq \sqrt{w^0}$, or $\sqrt{w^1} - \sqrt{w^0} \geq 1$. Going back one stage, the worker will accept if:

- $\sqrt{w^1} - \sqrt{w^0} \geq 1$ and $\sqrt{w^1} - 1 \geq 0$ (worker chooses $e = 1$)
- $\sqrt{w^1} - \sqrt{w^0} < 1$ and $\sqrt{w^0} \geq 0$ (worker chooses $e = 0$)

The owner's expected profit is:

- if $\sqrt{w^1} - \sqrt{w^0} \geq 1$ and $\sqrt{w^1} \geq 0$, the worker chooses $e = 1$, expected profit is $\frac{1}{3}(0 - w^1) + \frac{2}{3}(4 - w^1) = \frac{8}{3} - w^1$
- if $\sqrt{w^1} - \sqrt{w^0} < 1$ and $\sqrt{w^0} \geq 0$, the worker chooses $e = 0$, expected profit is $\frac{2}{3}(0 - w^0) + \frac{1}{3}(4 - w^0) = \frac{4}{3} - w^0$
- otherwise, the worker will reject, and expected profit is 0.

The increase in revenue from going from $e = 0$ to $e = 1$ is $\frac{4}{3}$, which is larger than the incremental cost of going from paying w^0 to w^1. Therefore, the profit-maximizing contract is to offer $w^0 = 0, w^1 = 1$. The worker will accept the contract and choose $e = 1$.

(b) Suppose e is not observable by the owner, so the contract specifies the wage conditional on the return r: w_0, w_4. In the last stage, the worker's expected utility is:

- If $e = 0$, $\frac{2}{3}(\sqrt{w_0} - 0) + \frac{1}{3}(\sqrt{w_4} - 0)$
- If $e = 1$, $\frac{1}{3}(\sqrt{w_0} - 1) + \frac{2}{3}(\sqrt{w_4} - 1)$

(c) The worker will choose $e = 1$ if $\frac{1}{3}(\sqrt{w_0} - 1) + \frac{2}{3}(\sqrt{w_4} - 1) \geq \frac{2}{3}(\sqrt{w_0} - 0) + \frac{1}{3}(\sqrt{w_4} - 0)$, or if $\sqrt{w_4} - \sqrt{w_0} \geq 3$. The worker will accept if:

- $\sqrt{w_4} - \sqrt{w_0} \geq 3$ and $\frac{1}{3}(\sqrt{w_0} - 1) + \frac{2}{3}(\sqrt{w_4} - 1) \geq 0$ (worker chooses $e = 1$)
- $\sqrt{w_4} - \sqrt{w_0} < 3$ and $\frac{2}{3}(\sqrt{w_0} - 0) + \frac{1}{3}(\sqrt{w_4} - 0) \geq 0$ (worker chooses $e = 0$)

The owner's expected profit is:

- if the worker chooses $e = 0$, expected profit is $\frac{2}{3}(0 - w_0) + \frac{1}{3}(4 - w_4)$
- if the worker chooses $e = 1$, expected profit is $\frac{1}{3}(0 - w_0) + \frac{2}{3}(4 - w_4)$

The difference in expected profit between $e = 0$ and $e = 1$ is $\frac{4+w_0-w_4}{3}$. In order to incentivize $e = 1$, $\sqrt{w_4} - \sqrt{w_0} \geq 3$. If w_0 is 0, then w_4 must be at least 9. This outweighs the increase in profit of $-\frac{5}{3}$, so the owner's profit maximizing contract is to offer $w_0 = 0, w_4 = 0$. The worker will accept the contract and choose $e = 0$.