Topics in Bank Management: Lecture 4

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Two-Period Economy

- By convention, we will say that $t = 0$ is state $s = 0$, and the states at $t = 1$ are $s = 1, 2, ..., S$.
- Let y^s denote the amount of consumption in state s.
- Assume agents have a utility function

$$v(y^0) + \delta E\left[v(y)\right] = v(y^0) + \delta \sum_{s=1}^{S} \pi_s v(y^s)$$

- pi_s is the probability that state s occurs.
- $v(\cdot)$ is a vNM utility function.
- $\delta \in (0, 1)$ is the discount factor.
- This type of utility function is *time-separable*, i.e. additive in the utility for $t = 0$ and $t = 1$.

Efficient Risk-Sharing

- Suppose there are two agents, $S = \{1, 2\}$, and agents are risk-averse: $v^i(\cdot)$ is strictly concave.
- Agents are endowed with some amount of securities that pay off at $t = 1$.
- Assume there is no *aggregate* risk: the sum of endowments for each state s is constant.
- There may be *idiosyncratic* risk: the endowment for an individual agent may differ across states.
- The *mutuality principle* states that an efficient allocation in this situation will diversify away idiosyncratic risk.
- Agents will consume the same amount in both states; they will only bear aggregate risk.

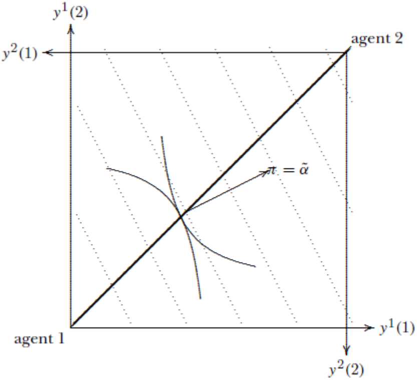

Figure 5.1. An Edgeworth box with no aggregate risk and full insurance. The dotted lines are iso-expected wealth lines, the fat line is the contract curve.

- Assume each agent's utility is:

$$u_i(y^0, y^1, y^2) = v(y^0(i)) + \delta \sum_{s=1}^{S} \pi_s v(y^s(i))$$

- $y^s(i)$ is the amount consumed in state s by agent i.
- Assume the same aggregate income in both states: $y^1(1) + y^1(2) = y^2(1) + y^2(2) = W$
- At equilibrium, both agents' MRS are equal to each other and the price ratio.

$$\frac{\frac{\partial u_1}{\partial y^1(1)}}{\frac{\partial u_1}{\partial y^2(1)}} = \frac{\frac{\partial u_2}{\partial y^1(2)}}{\frac{\partial u_2}{\partial y^2(2)}}$$

$$\frac{\pi_1 v_1'(y^1(1))}{\pi_2 v_1'(y^2(1))} = \frac{\pi_1 v_2'(W - y^1(1))}{\pi_2 v_2'(W - y^2(1))}$$

$$\frac{v_1'(y^1(1))}{v_2'(W - y^1(1))} = \frac{v_1'(y^2(1))}{v_2'(W - y^2(1))}$$

$$\frac{\frac{\partial u_1}{\partial y^1(1)}}{\frac{\partial u_1}{\partial y^2(1)}} = \frac{\frac{\partial u_2}{\partial y^1(2)}}{\frac{\partial u_2}{\partial y^2(2)}}$$

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$$\frac{v_1'(y^1(1))}{v_2'(W - y^1(1))} = \frac{v_1'(y^2(1))}{v_2'(W - y^2(1))}$$

- By assumption, v_1, v_2 are strictly concave, therefore v_1', v_2' are strictly decreasing.
- The function $f(x) = \frac{v_1'(x)}{v_2'(W-x)}$ is strictly decreasing, so if two values x, x' give $f(x) = f(x')$, then $x = x'$.
- Therefore, $y^1(1) = y^2(1)$ and both agents consume the same amount in each state.

Mutuality Principle

- **Lengwiler Box 5.1 (Mutuality Principle)**: An efficient allocation of risk requires that only aggregate risk be borne by agents. All idiosyncratic risk is diversified away by mutual insurance among agents.
- The marginal aggregate risk borne by an agent equals the ratio of his absolute risk tolerance to the average risk tolerance of the population.
- The mutuality principle can fail if:
  - Beliefs are heterogeneous (different agents have different subjective probabilities of states)
  - if market frictions (e.g. trading costs, short sale constraints) impede Pareto efficiency
  - if markets are incomplete (we'll get to this in a few minutes)

Mutuality Principle

- This principle has many applications in different fields of economics.
  - In international macro, many papers try to test efficient risk sharing among different countries, and explain if/why it does not occur
  - In labor, test efficient risk sharing among workers, retirees, health insurance consumers, etc
- Many of the models we have seen in this course want to explain banks as a way to implement some sort of risk-sharing.
- However, risk-sharing is not the only motivation for financial transactions.

Complete & Incomplete Asset Markets

- Let's go back to the *asset economy*, where a security is described by the vector of returns for each possible state.
- Asset j is specified by: $r^j = (r^j_1, ..., r^j_S)^T$
- Whoever holds 1 unit of asset j will receive r^j_s at $t = 1$, if the state of the world happens to be s.
- A *storage asset* (e.g. cash) would be $(1, ..., 1)^T$.
- A *riskless bond* with nominal yield $1 + r$ would be $(1 + r, ..., 1 + r)^T$.
- An *Arrow security* for state s is $e^s = (0, 0, ..., 1, ...0)^T$

Complete & Incomplete Asset Markets

- Suppose that there are a total of J securities traded on the market.
- We can collect the return vector for each asset into a $S \times J$ matrix:

$$\begin{pmatrix} r_1^1 & \dots & r_1^J \\ \vdots & \ddots & \vdots \\ r_S^1 & \dots & r_S^J \end{pmatrix}$$

- Suppose that instead of trading contingent claims, agents can only trade the assets specified by this matrix.
- Is it possible for this asset-only market to achieve the same equilibrium allocations as the contingent-claim markets?
- The answer is yes, if the markets are *complete*: if it is possible to insure each state separately.
- That is, it is possible to affect the payoff in one specific state without affecting the payoffs in other states.

Complete & Incomplete Asset Markets

- Suppose the matrix of available assets is given by

$$\begin{pmatrix} r_1^1 & \cdots & r_1^J \\ \vdots & \ddots & \vdots \\ r_S^1 & \cdots & r_S^J \end{pmatrix}$$

- Markets are complete if this matrix is of rank S.
- From linear algebra, we know that in a full column-rank matrix, any vector can be represented as a linear combination of the columns in the matrix.

Complete & Incomplete Asset Markets

- The matrix of S Arrow securities is the identity matrix. It is obviously of full rank, and therefore complete.

$$\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

- If we removed one Arrow security, the markets would become incomplete.
- It would not be possible to insure that state, since no combination of the other assets have a payoff in that state alone.

Effects of Incomplete Markets

- **Lengwiler Box 3.14 (Effects of incomplete markets)**:
  - Arrow prices associated with an equilibrium are not unique;
  - typically, an equilibrium is not Pareto efficient (e.g. mutuality fails)
- What does market incompleteness mean in the real world?
- If there is no security or insurance contract to hedge against a specific event.
- In theory, financial innovation and the creation of new types of securities (credit swaps, derivatives, CDS, etc) should decrease incompleteness.
- For example, if a corporation has a credit default swap traded on it, it is possible to hedge against the bankruptcy of that corporation.
- However, as the financial crisis demonstrated, the issuers or counterparties to these securities may also fail (e.g. AIG).
- It may never be possible for the markets to become truly complete.

The Risk-Sharing Approach to Borrower-Lender Relationship

- As we've seen, risk-sharing is one way to view a trade of contingent claims between two agents.
- Suppose there are two agents: a borrower/entrepreneur, and a lender/investor.
- Two periods: $t = 0, 1$ and one good.
- At $t = 0$, the borrower has a project that requires investment I, and will produce a random return $\tilde{y}$ at $t = 1$.
- Assume the borrower has no resources, and has to borrow the entire amount I.
- Both agents consume only at $t = 1$ and have vNM utility functions u_L, u_B, assumed to be twice differentiable, concave, and increasing.

Symmetric Information Case

- Suppose $\tilde{y}$ is observable by both agents (symmetric information).
- The agents can sign a contract specifying how to share the payout $\tilde{y}$ at $t = 1$.
- The contract will specify an amount to be repaid to the lender for every possible value of $\tilde{y}$
- This is the *repayment* function, $R(y)$.
- The borrower will keep the remainder, $y - R(y)$.
- An optimal debt contract is a solution to the following problem:

$$\max_{R(\cdot)} E\left[u_B(\tilde{y} - R(\tilde{y}))\right] \qquad \text{subject to} \tag{1}$$

$$E\left[u_L(R(\tilde{y}))\right] \geq U_L^0 \tag{2}$$

$$\max_{R(\cdot)} E\left[u_B(\tilde{y} - R(\tilde{y}))\right] \qquad \text{subject to} \tag{3}$$

$$E\left[u_L(R(\tilde{y}))\right] \geq U_L^0 \tag{4}$$

- The constraint on the lender's expected utility is called the *individual rationality constraint*.
- U_L^0 is the lender's reservation utility, i.e. the amount of utility he could get elsewhere.
- The contract must provide at least U_L^0 in expectation, for it to be optimal for the lender to participate.
- If the support of $\tilde{y}$ is finite (i.e. it can take a finite number of values), then this problem has a finite number of variables.
- We can also restate the problem as maximizing the lender's expected utility, subject to a individual rationality constraint on the borrower.
- Since u_B and u_L are increasing by assumption, the constraint is always binding.

- Suppose $\tilde{y}$ can take on values $y_1, ..., y_K$ with probabilities $\pi_1, ..., \pi_K$.
- Let $R_1, ...R_K$ denote the repayment level $R(y_1), ..., R(y_K)$. The problem becomes:

$$\max_{R_1 \ldots R_K} \sum_{i=1}^{K} \pi_i u_B(y_i - R_i) \qquad \text{subject to}$$

$$\sum_{i=1}^{K} \pi_i u_L(R_i) \geq U_L^0$$

- Lagrangian:

$$L(R_1, ..., R_K, \mu) = \sum_{i=1}^{K} \pi_i u_B(y_i - R_i) - \mu(U_L^0 - \sum_{i=1}^{K} \pi_i u_L(R_i))$$

$$\frac{\partial L}{\partial R_i} = -\pi_i u_B'(y_i - R_i) + \mu \pi_i u_L'(R_i) = 0 \qquad \text{for all } i$$

$$\frac{\partial L}{\partial \mu} = U_L^0 - \sum_{i=1}^{K} \pi_i u_L(R_i) = 0$$

$$\frac{u_B'(y_i - R_i)}{u_L'(R_i)} = \mu \qquad \text{for all } i$$

- For any i, j, the ratio of marginal utilities is a constant:

$$\frac{u'_B(y_i - R_i)}{u'_B(y_j - R_j)} = \frac{u'_L(R_i)}{u'_L(R_j)}$$

- Let's go back to assuming that $\tilde{y}$ can take on any value on an interval.
- Take logs of the equation, then differentiate with respect to y:

$$\frac{u'_B(y - R(y))}{u'_L(R(y))} = \mu$$

$$\ln(u'B(y - R(y)) - \ln(u'_L(R(y))) = \ln(\mu)$$

$$\frac{u''_B(y - R(y))}{u'_B(y - R(y))}(1 - R'(y)) - \frac{u''_L(R(y))}{u'_L(R(y))}R'(y) = 0$$

$$I_B(y - R(y))(1 - R'(y)) + I_L(R(y))R'(y) = 0$$

$$R'(y) = \frac{I_B(y - R(y))}{I_B(y - R(y)) + I_L(R(y))}$$

- where I_B, I_L are the coefficients of absolute risk aversion.

$$R'(y) = \frac{I_B(y - R(y))}{I_B(y - R(y)) + I_L(R(y))}$$

- This equation says that the sensitivity of the repayment function $R(y)$ to y increases as the risk aversion of the borrower relative to the lender increases (I_B/I_L increases).
- Suppose there were two possible outcomes for $\tilde{y}$, $y_1 < y_2$.
- If the borrower were risk-neutral ($I_B = 0$), then $R_2 - R_1 = 0$.
- As the borrower becomes more risk-averse, $R_2 - R_1$ increases.
- The borrower is transferring consumption from the high state y_2 to the low state y_1: repay more in the high state, repay less in the low state.
- Compare this situation to the GE problem earlier: here, there is aggregate uncertainty, since the aggregate income y varies in different states of the world.
- Therefore, the agents will not achieve perfect insurance.
- If the lender is risk-neutral ($I_L = 0$), then the borrower can achieve perfect insurance.

- These risk-sharing results are exactly what we would get from the solution of a general equilibrium problem.
- One approach to modeling banks is to view banks as simply a way of implementing this sort of risk-sharing transaction.
- However, in the real world, banks have many characteristics that do not match the GE solution.
  - Banks have large, diversified portfolios, so they should be risk-neutral with respect to a single borrower.
  - This implies that $R'(y)$ should be close to 1 (i.e. perfect insurance for the borrower).
  - However, real bank loans typically have a *constant repayment*, i.e. $R(y) = \bar{R}$, no matter what the outcome is.
- Therefore, we will drop the assumption of symmetric information, and suppose instead that y is costly to observe.

Costly State Verification

- Based on Townsend (1979), Gale & Hellwig (1985)
- Assume the realization y of $\tilde{y}$ is *not observable* by lender, unless they undertake an *audit*, which costs γ.
- Suppose the borrower reports that the project's cash flow is $\hat{y}$, which may not be truthful.
- A contract between the borrower and lender must now specify:
  - A repayment function $\hat{y} \to R(\hat{y})$
  - An *auditing rule* A. This is a set of reported cashflows $\hat{y}$ for which the lender will undertake an audit.
  - A penalty/reward function $P(y, \hat{y})$ that specifies a transfer between the borrower and lender if an audit takes place after the borrower reports $\hat{y}$ and the audit reveals the true cashflow y.
- A contract is a triple $(R(\cdot), A, P(\cdot, \cdot))$.

Incentive Compatibility

- We will assume that contracts must satisfy an *incentive compatibility* constraint: it must be optimal for the borrower to tell the truth, i.e. report $\hat{y} = y$ in all states of the world.
- Also assume *limited liability*: consumption for either agent cannot be negative. This implies $0 \leq R(y) \leq y$ for all y.
- For incentive compatibility, we can set the penalty function $P(y, \hat{y})$ to be arbitrarily large whenever the audit reveals the borrower did not tell the truth, i.e. when $\hat{y} \neq y$.
- Also, we can set the penalty to zero whenever the audit reveals the borrower did tell the truth, $\hat{y} = y$.
- This penalty function will make it optimal to tell the truth whenever $y \in A$, but does not ensure truthfulness if $y \notin A$.

- How can we ensure there is no incentive to lie if y is outside the audit region A? This will depend on the repayment function $R(\hat{y})$.
- The repayment function must be *constant* if $y \notin A$, otherwise there is an incentive to report a $\hat{y}$ that results in the smallest possible repayment.
- Denote this constant amount by R. Then $R(\hat{y}) = R$ for $\hat{y} \notin A$.
- To remove an incentive to lie when $y \in A$, R must be at least as large as the maximum payment possible on A.
- Otherwise, for some $y \in A$, the borrower can reduce his repayment to R by untruthfully reporting a $\hat{y}$ outside A.
- **Result 4.2 (a)**: A contract is incentive compatible if and only if there exists a repayment level R such that:
  - For all $y \notin A$, $R(y) = R$
  - For all $y \in A$, $R(y) \leq R$

- **Result 4.2 (a)**: A contract is incentive compatible if and only if there exists a repayment level R such that:
  - For all $y \notin A$, $R(y) = R$
  - For all $y \in A$, $R(y) \leq R$
- The previous slide proved that incentive compatibility implies these conditions. Let's prove the converse.
- Case 1: Suppose $y \in A$. If the borrower reports $\hat{y} \neq y$ and:
  - $\hat{y} \in A$, there will be punishment.
  - $\hat{y} \notin A$, the borrower must repay $R \geq R(y)$. Therefore, there is no incentive to be untruthful.
- Case 2: Suppose $y \notin A$. If the borrower reports $\hat{y} \neq y$ and:
  - $\hat{y} \in A$, there will be punishment.
  - $\hat{y} \notin A$, the borrower must repay the same amount R. Therefore, there is no incentive to be untruthful.

- Among the class of incentive compatible contracts, which ones are efficient (i.e. Pareto-optimal)?
- Assume that both agents are risk-neutral, so only the expected value of payoffs matter.
- An efficient contract is one that *minimizes* the probability of an audit (and therefore expected punishment), subject to a given level of expected repayment to the lender.
- Equivalently, an efficient contract *maximizes* the expected repayment to the lender, subject to a given probability of an audit occurring.
- In either case, it is not possible to make one agent better off without making the other worse off.
- A *standard debt contract* is a contract specifying some fixed repayment R, where the borrower repays (depending on cashflow y):
  - If $y \leq R$, borrower repays y.
  - If $y > R$, borrower repays R.

- An incentive-compatible contract $(R^*(\cdot), A^*,)$ is efficient iff:
  - for all $y \imath A^*$, $R^(y) = \min(y, R^*)$. That is, the borrower will repay as much of R as possible, subject to limited liability.
  - $A^* = \{y | y < R^*\}$. An audit will take place only when the reported cashflow is less than R^*.
- This can be interpreted as a standard debt contract.
- **Result 4.2 (b)**: If both agents are risk-neutral, any efficient, incentive-compatible contract is a standard debt contract.

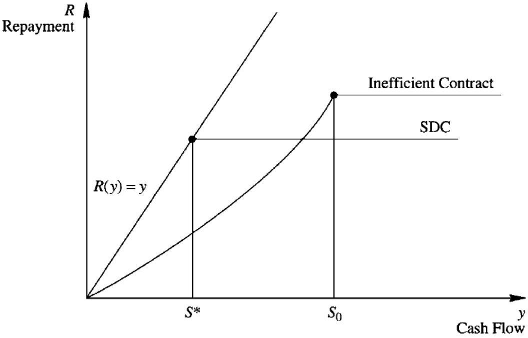

Figure 4.1
Optimality of the standard debt contract under costly state verification.

- This graph compares two contracts that have the same expected repayment to the lender.
- The first one, SDC, repays all of y up to $y = S*$, then repays S^* for $y > S^*$.
- The second contract pays less (the curved line) up to S_0, then pays S_0 for $y > S_0$.

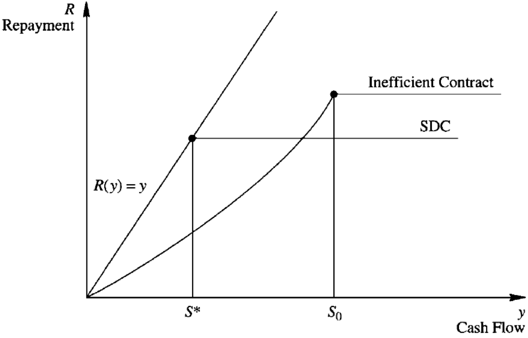

Figure 4.1
Optimality of the standard debt contract under costly state verification.

- The audit region for SDC is smaller than the audit region for the alternative contract.

Summary

- One way to look at a bank loan is as a form of risk-sharing.
- The GE solution tells us that at a Pareto-efficient allocation:
  - Risk-averse agents will diversify away all idiosyncratic risk.
  - Agents with lower risk aversion will bear more aggregate risk.
- However, bank loans in the real world don't have the characteristics of a risk-sharing arrangement.
- If we assume that the lender cannot observe the true cashflow (a form of asymmetric information), then we can show that the "standard debt contract" is efficient and incentive-compatible.

Next Week

- For next week, please read Ch. 4.4, 4.5-4.5.1, and 4.6 in Freixas & Rochet.