CUR 412: Game Theory and its Applications, Lecture 12

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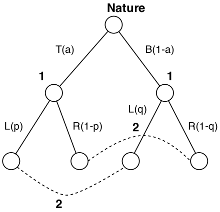
Announcements

- Homework #4 is due next week.

Review of Last Lecture

- In extensive games with imperfect information, players are uncertain about what actions have been chosen in the past.
- A player knows that he is at a particular *information set* (i.e. a set of histories), but does not know exactly which history he is at.
- A player still has *beliefs* about which history in his information set is more likely. We use a probability distribution to model beliefs.
- We want a player's belief at every information set to match the true probability distribution, given the behavioral strategies of all players.
- That is, a player's belief should match the conditional probabilities of the histories, conditional on reaching the information set.
- If an information set is reached with probability 0, any beliefs are consistent.

Example 3



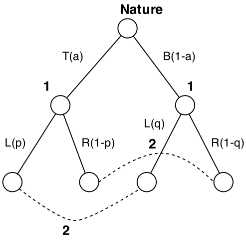
- Suppose we are at information set $\{TL, BL\}$. What is the probability that history TL has occurred?

$$P(TL|\{TL, BL\}) = \frac{P(TL \cap \{TL, BL\})}{P(\{TL, BL\})} = \frac{P(TL)}{P(\{TL, BL\})}$$
$$= \frac{ap}{ap + (1-a)q}$$

- Therefore, the beliefs at information set $\{TL, BL\}$ that would be consistent with the behavioral strategy of Player 1 are:

$$\left(\frac{ap}{ap + (1-a)q}, \frac{(1-a)q}{ap + (1-a)q}\right)$$

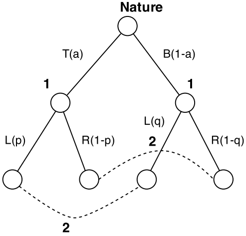
Example 3



- Similarly, the beliefs at information set $\{TR, BR\}$ that would be consistent with the behavioral strategy of Player 1 are:

$$\left(\frac{a(1-p)}{a(1-p)+(1-a)(1-q)}, \frac{(1-a)(1-q)}{a(1-p)+(1-a)(1-q)}\right)$$

Example 3



- Information set $\{TL, BL\}$ is reached with zero probability if $p = 0, q = 0$.
- Information set $\{TR, BR\}$ is reached with zero probability if $p = 1, q = 1$.
- Any probability distribution is a consistent belief at an information set reached with zero probability.

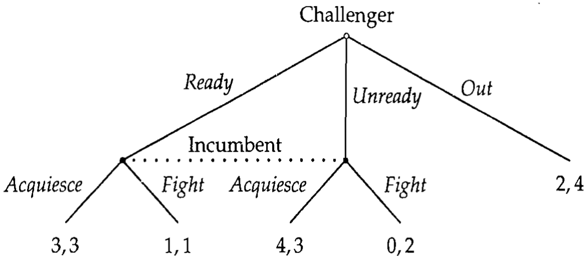
Belief System

- Now, we know how to calculate beliefs at all information sets, which should allow us to calculate expected payoffs for each player's action.
- Let's state some definitions which will lead to our concept of equilibrium:
- **Definition 324.1**: A **belief system** is a function that assigns to each information set, a probability distribution over the histories in that information set.
- A belief system is simply a list of beliefs for every information set.

Weak Sequential Equilibrium

- **Definition 325.1**: An **assessment** is a pair consisting of a profile of behavioral strategies, and a belief system. An assessment is called a **weak sequential equilibrium** if these two conditions are satisfied:
  - All players' strategies are optimal when they have to move, given their beliefs and the other players' strategies. This is called the *sequential rationality* condition.
  - All players' beliefs are consistent with the strategy profile, i.e. their beliefs are what is computed by Bayes' Rule for information sets that are reached with positive probability. This is called the *consistency of beliefs* condition.
- Note that this generalizes the two conditions of Nash equilibrium.
- Clearly, this may be difficult to find for complex games. We will usually simplify things by assuming that players only use pure strategies (Nature still randomizes).
- To specify a weak sequential equilibrium, we need two things:
  - The strategy profile (actions at every information set);
  - The beliefs at every information set.

Example 317.1: Entry Game with Preparation

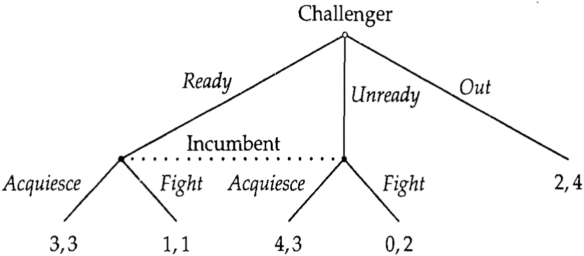


- Consider a modified entry game. The challenger has three choices: stay out, prepare for a fight and enter, or enter without preparation.
- Preparation is costly, but reduces the loss from a fight.
- As before, the incumbent can *Fight* or *Acquiesce*.
- A fight is less costly to the incumbent if the entrant is unprepared, but the incumbent still prefers to *Acquiesce*.
- *Incumbent* does not know if *Challenger* is prepared or unprepared.

Example 317.1: Entry Game with Preparation

- First, let's convert this game to strategic form and find the NE.
- Then, we will see which NE can also be part of a weak sequential equilibrium.
- For each NE, we will find the beliefs that are consistent with the strategy profile.
- Then, check if the strategies are optimal, given beliefs.
- If both conditions are satisfied, then we have found a weak sequential equilibrium.

Strategic Form



	Acquiesce	*Fight*
Ready	3,3	1,1
Unready	4,3	0,2
Out	2,4	2,4

	Acquiesce	*Fight*
Ready	3,3	1,1
Unready	4,3	0,2
Out	2,4	2,4

- NE are: $(Unready, Acquiesce)$ and $(Out, Fight)$.
- Consider $(Unready, Acquiesce)$: at Player 2's information set, only consistent belief over $\{Ready, Unready\}$ is $(0, 1)$.
  - Given this belief, the optimal action for Player 2 is *Acquiesce*. This matches the NE, so this is a WSE.
- Consider $(Out, Fight)$: Player 2's information set is not reached, so any beliefs over $\{Ready, Unready\}$ are consistent.
  - However, *Acquiesce* is optimal given any belief over $\{Ready, Unready\}$. Therefore, this NE cannot be a WSE.

Adverse Selection

- Let's examine one more application of imperfect information.
- Suppose Player 1 owns a car, and Player 2 is considering whether to buy the car from Player 1.
- The car has a level of quality α, which can take on three types, L, M, H.
- Player 1 knows the type of the car, but Player 2 does not.
- Player 1's valuation of the car is:

$$v_1(\alpha) = \begin{cases} 10 & \text{if } \alpha = L \\ 20 & \text{if } \alpha = M \\ 30 & \text{if } \alpha = H \end{cases}$$

- The higher the quality, the higher is Player 1's valuation of the car.

Adverse Selection

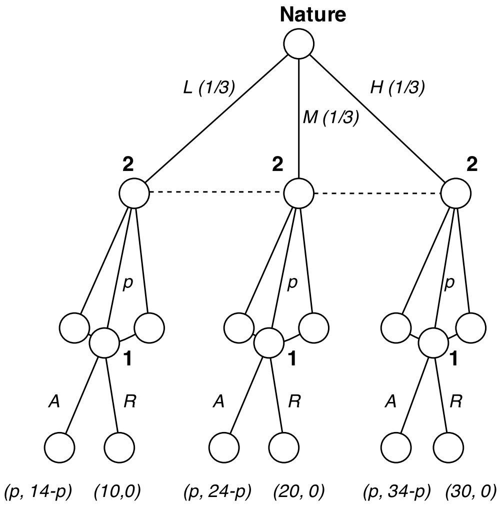
- Likewise, suppose that Player 2 has a similar valuation of the car:

$$v_2(\alpha) = \begin{cases} 14 & \text{if } \alpha = L \\ 24 & \text{if } \alpha = M \\ 34 & \text{if } \alpha = H \end{cases}$$

- Player 2 knows that the probability distribution of quality levels in the general population is $\frac{1}{3}$ for each type.
- Note that Player 2's valuation of the car is higher than Player 1's valuation, for all quality levels of the car.
- Therefore, if the type were common knowledge, a trade should occur: it is always possible to find a Pareto-efficient trade (that is, both players are not worse off, and at least one player is better off).
- For example, if it were known that quality was L, a trade at a price between 10 and 14 would make both players better off.

Adverse Selection

- Consider the following game:
  - Nature chooses the type of the car with $P(L) = P(M) = P(H) = \frac{1}{3}$. Player 1 observes the type; Player 2 does not.
  - Player 2 makes a price offer $p \geq 0$ to Player 1 for the car.
  - Player 1 can accept (A) or reject (R).
  - If Player 1 accepts, he gets the price offered, and the car is transferred to Player 2.
  - If trade occurs, Player 1's payoff is the price offered. Player 2's payoff is his valuation of the car, minus the price paid.
  - If trade does not occur, Player 1's payoff is his valuation of the car. Player 2's payoff is zero.


Nature
L (1/3)
M (1/3)
H (1/3)
2
2
2
p
p
p
1
1
1
A
R
A
R
A
R
(p, 14-p)
(10,0)
(p, 24-p)
(20, 0)
(p, 34-p)
(30, 0)

Adverse Selection

- Consider the subgame after p has been offered.
- Player 1's best response is:
  - If $\alpha = L$, accept if $p \geq 10$, reject otherwise.
  - If $\alpha = M$, accept if $p \geq 20$, reject otherwise.
  - If $\alpha = H$, accept if $p \geq 30$, reject otherwise.
- Now, consider Player 2's decision. His beliefs match Nature's probability distribution: the probability on L, M, H is $1/3$ each.
- Let's find the expected payoff $E_2(p)$ of choosing p, for the range $p \geq 0$.

Adverse Selection

- If $p < 10$, Player 1 will reject in all cases. $E_2(p) = 0$.
- If $10 \le p \le 14$, $E_2(p) = \frac{1}{3}(14 - p) + \frac{1}{3}(0) + \frac{1}{3}(0) = \frac{14-p}{3}$.
  - This is non-negative if p is in this range.
- If $14 < p < 20$, $E_2(p) = \frac{1}{3}(14 - p) + \frac{1}{3}(0) + \frac{1}{3}(0) = \frac{14-p}{3} < 0$.
- If $20 \le p \le 24$, $E_2(p) = \frac{1}{3}(14 - p) + \frac{1}{3}(24 - p) + \frac{1}{3}(0) = \frac{38-2p}{3} < 0$.
- If $24 < p < 30$, $E_2(p) = \frac{1}{3}(14 - p) + \frac{1}{3}(24 - p) + \frac{1}{3}(0) = \frac{38-2p}{3} < 0$.
- If $30 \le p$, $E_2(p) = \frac{1}{3}(14 - p) + \frac{1}{3}(24 - p) + \frac{1}{3}(34 - p) = \frac{72-3p}{3} < 0$.

Adverse Selection

- The optimal choice is for Player 2 to offer $p = 10$. If $\alpha = L$, Player 1 will accept; otherwise Player 1 will reject.
- Note that in only the lowest quality case does trade occur.
- This is clearly inefficient, since trades that would benefit both parties are not taking place.
- This is an example of *adverse selection*, in which the low-quality type "drives out" the high-quality type from the market, due to uncertainty.
- One way to overcome this problem is if the buyer could get some information about the true quality of the car.
- The seller could simply say that the car is high-quality. But why should the buyer believe him?
- We want to know if there is a method by which the seller can credibly communicate the quality of the car.

Ch. 10.5: Signaling Games

- In many situations, Player 1 may know something that Player 2 does not, which would affect Player 2's choice if he knew. For example:
  - When you buy an item from a seller, the seller knows the item's quality, but you do not.
  - When a firm hires an employee, the employee knows his skill level, but the firm does not.
  - When two people get into a competition, each person knows how strong he is, but the other person does not.
- This is called a situation with *asymmetric information*.

- We can model this kind of situation by assuming there are two (or more) *types* of Player 1, which is chosen by Nature with a known probability distribution.
- Suppose there are two types of Player 1, the "high" type H, and the "low" type L. H-types are more preferable to Player 2.
- Supose Player 1 is a H-type. Then he would like to somehow let Player 2 know that he is a H-type.
- On the other hand, suppose Player 2 is a L-type. Then he would like to imitate the H-type, and make Player 2 believe that he is type H.
- A H-type Player 1 could simply say that he is H-type, but why should Player 2 believe it? A L-type player could do exactly the same thing.
- However, if there was some test that L-types found more costly to pass than H-types, then perhaps L-types would rationally choose not to take the test at all.

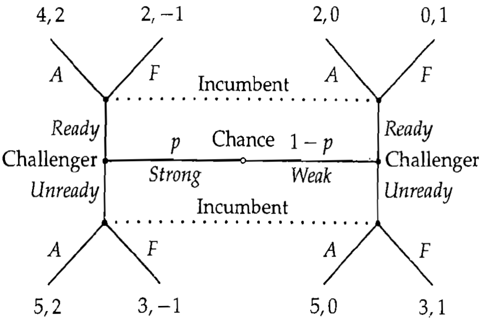
Ch. 10.5: Signaling Games

- These games are called *signaling games*.
- At the beginning, Nature chooses Player 1's type according to a known distribution.
- Player 1 can choose to send a *costly signal* that Player 2 observes.
- Player 2 then chooses his action, taking Player 1's choice into account.
- In a *pooling equilibrium*, H and L types of Player 1 behave the same way, and Player 2 cannot distinguish between them.
- In a *separating equilibrium*, H and L types behave differently, and Player 2 can tell them apart.

Entry Game with Signaling

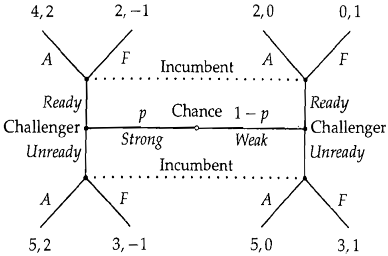
- Consider this variant of the Entry Game.
- There are two types of *Challenger*, *Strong* and *Weak*, which occur with probabilities p and $1 - p$.
- The *Challenger* knows his type, but the *Incumbent* does not.
- *Challenger* can choose to be *Ready* or *Unready* for competition. Choosing *Ready* is more costly for a *Weak*-type.
- The *Incumbent* can choose to fight, F, or acquiesce, A.
- *Incumbent* prefers to fight a *Weak* challenger, but prefers to acquiesce to a *Strong* challenger.
- If the *Incumbent* chooses to fight, both types of challenger suffer the same cost.

Entry Game with Signaling



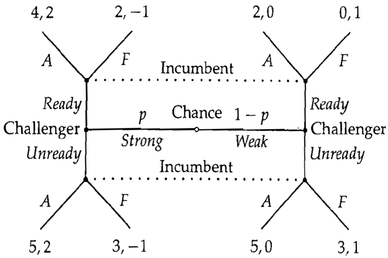
- The start of the game is in the center.
- The first number is the *Challenger*'s payoff. *Challenger*'s payoff is 5 if *Incumbent* chooses *A*.
- Cost of *Ready* is 1 for *Strong*-type, 3 for *Weak*-type.

Entry Game with Signaling



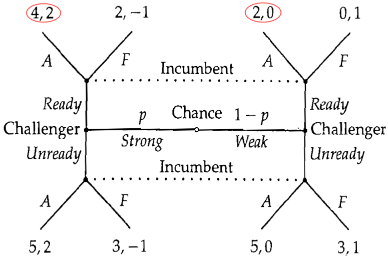
- *Challenger* has two information sets: $\{Strong\}$ and $\{Weak\}$.
- *Incumbent* has two information sets: $\{(Strong, Ready), (Weak, Ready)\}$ and $\{(Strong, Unready), (Weak, Unready)\}$.
- Each player has 2 actions at each information set, so a total of 4 pure strategies.

Entry Game with Signaling



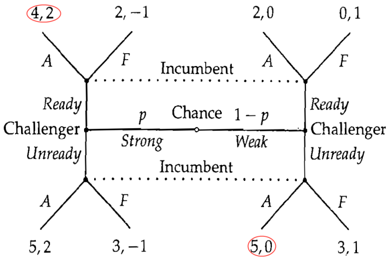
- *Challenger*'s pure strategies: RR, RU, UR, UU
- *Incumbent*'s pure strategies: $(A|R, A|U), (A|R, F|U), (F|R, A|U), (F|R, F|U)$ where the notation $(A|R, A|U)$ means: choose A conditional on R, choose A conditional on U
- Suppose $0 < p < 1$, so both of *Incumbent*'s information sets are reached with positive probability. Let's calculate the expected payoffs to pure strategy profiles.

Entry Game with Signaling



- For the strategy profile $(RR, (A|R, A|U))$, the outcome will be $(Strong, R, A)$ with probability p, and $(Weak, R, A)$ with probability $(1 - p)$.
- Expected payoffs are $(4p + 2(1 - p), 2p)$.

Entry Game with Signaling



- For the strategy profile $(RU, (A|R, A|U))$, the outcome will be $(Strong, R, A)$ with probability p, and $(Weak, U, A)$ with probability $(1 - p)$.
- Expected payoffs are $(4p + 5(1 - p), 2p)$.

Expected Payoffs

	RR	RU
$A\|R, A\|U$	p(4,2) + (1-p)(2,0)	p(4,2) + (1-p)(5,0)
$A\|R, F\|U$	p(4,2) + (1-p)(2,0)	p(4,2) + (1-p)(3,1)
$F\|R, A\|U$	p(2,-1) + (1-p)(0,1)	p(2,-1) + (1-p)(5,0)
$F\|R, F\|U$	p(2,-1) + (1-p)(0,1)	p(2,-1) + (1-p)(3,1)

	UR	UU
$A\|R, A\|U$	p(5,2) + (1-p)(2,0)	p(5,2) + (1-p)(5,0)
$A\|R, F\|U$	p(3,-1) + (1-p)(2,0)	p(3,-1) + (1-p)(3,1)
$F\|R, A\|U$	p(5,2) + (1-p)(0,1)	p(5,2) + (1-p)(5,0)
$F\|R, F\|U$	p(3,-1) + (1-p)(0,1)	p(3,-1) + (1-p)(3,1)

Expected Payoffs with $p = 0.5$

	$A\|R, A\|U$	$A\|R, F\|U$	$F\|R, A\|U$	$F\|R, F\|U$
RR	3,1	3,1	1,0	1,0
RU	4.5, 1	3.5, 1.5	3.5, -0.5	2.5, 0
UR	3.5, 1	2.5, -0.5	2.5, 1.5	1.5, 0
UU	5, 1	3, 0	5, 1	3, 0

- The NE are: $(RU, (A|R, F|U)), (UU, (A|R, A|U))$, and $(UU, F|R, A|U))$. Let's go through each of these and see which ones can be part of a WSE.

Check if NE is a WSE

- Consider the NE $(RU, (A|R, F|U))$.
- The beliefs at *Incumbent*'s information set after R must be $(1, 0)$ (probability 1 on *Strong*) since *Challenger*'s strategy will only choose R if Nature chose *Strong*.
- The beliefs at *Incumbent*'s information set after U must be $(0, 1)$ (probability 0 on *Strong*).
- The actions specified for *Incumbent* at this NE are clearly optimal given these beliefs, so this is a WSE.

Check if NE is a WSE

- Consider the NE $(UU, (A|R, A|U))$.
- The beliefs at *Incumbent*'s information set after U must be $(p, 1-p) = (0.5, 0.5)$.
- Clearly, *Challenger*'s actions specified by this NE are optimal at this information set, since it is a NE.
- The information set after R is reached with zero probability, so any beliefs are consistent.
- Denote the beliefs at this information set as $(q, 1-q)$, where q is the probability on *Strong*.
- We want to find the range of q that makes A optimal, i.e. $E(A) \geq E(F)$, which is true if:

$$q2 + (1-q)0 \geq q(-1) + (1-q)1$$

- or if $q \geq \frac{1}{4}$.

Check if NE is a WSE

- Consider the NE $(UU, F|R, A|U)$.
- For the information set after U, everything is the same as in the previous case.
- The information set after R is reached with zero probability, so any beliefs are consistent.
- Denote the beliefs at this information set as $(q, 1-q)$, where q is the probability on *Strong*.
- We want to find the range of q that makes F optimal, i.e. $E(A) \leq E(F)$, which is true if:

$$q2 + (1-q)0 \leq q(-1) + (1-q)1$$

- or if $q \leq \frac{1}{4}$.

Separating & Pooling Equilibria

- There is one *separating* equilibrium: $(RU, (A|R, F|U))$, and *Incumbent*'s beliefs are $(1, 0)$ after R and $(0, 1)$ after U.
- In a separating equilibrium the different types of Player 1 choose different actions, so Player 2 knows which type they are by their choice.
- There is a set of *pooling* equilibria: $(UU, (A|R, A|U))$ and $(UU, F|R, A|U))$.
- In both of these cases, the information set after R is reached with zero probability, so any beliefs are consistent.
- If the beliefs are such that A to be optimal, then the WSE is $(UU, (A|R, A|U))$; otherwise, it is $(UU, F|R, A|U))$.
- In a pooling equilibrium, the different types of Player 1 behave the same way, and Player 2 cannot distinguish them.

Ch. 10.7: Education as a signal of ability

- Why do students obtain a college degree?
- One reason is that the knowledge they gain in college will increase their skills and abilities.
- However, there is another possible reason: perhaps students use degrees to *differentiate* themselves from other students when applying for jobs.
- This can hold even if the degree itself does not increase ability.
- We model this as a signaling game.

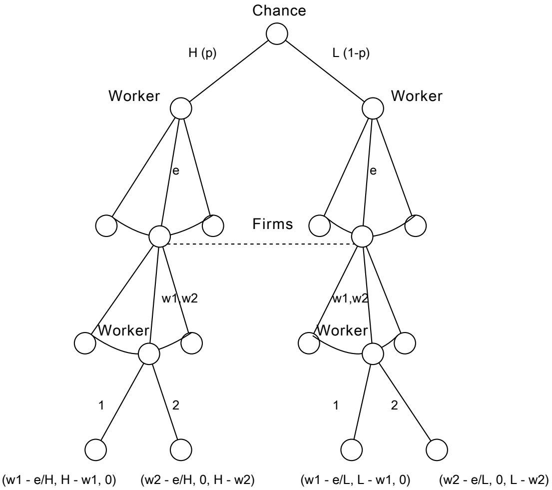
Ch. 10.7: Education as a signal of ability

- Suppose the ability level of a worker can be measured by a single number
- There are two types of workers: "high" and "low"-ability workers, denoted H and L, with $L < H$.
- Type is known to the workers, but cannot be directly observed by employers.
- Workers can choose to obtain some amount of education, which has no effect on ability, but costs less for the H-type worker.

Ch. 10.7: Education as a signal of ability

- The sequence of the game is as follows:
    - Chance chooses the type of the worker at random; the probability of H is p.
    - The worker, who knows his type, chooses an amount of education $e \geq 0$. The cost of education is different according to type; for a L-type worker, the cost is e/L; for a H-type worker, it is e/H.
    - Two firms observe the worker's choice of e (but not his type), and simultaneously offer two wages, w_1 and w_2.
    - The worker chooses one of the wage offers and works for that firm. The worker's payoff is his wage minus the cost of education. The firm that hires the worker gets a payoff of the worker's ability, minus the wage. The other firm gets a payoff of 0.

Game Tree


Chance
H (p)
L (1-p)
Worker
Worker
e
e
Firms
w1,w2
w1,w2
Worker
Worker
1
2
1
2
(w1 - e/H, H - w1, 0)
(w2 - e/H, 0, H - w2)
(w1 - e/L, L - w1, 0)
(w2 - e/L, 0, L - w2)

Finding WSE

- We claim there is a weak sequential equilibrium in which a H-type worker chooses a positive amount of education, and a L-type worker chooses zero education.
- Consider this assessment (i.e. beliefs plus strategies), where e^* is a positive number (to be determined):
  - *Worker's strategy*: Type H chooses $e = e^*$ and type L chooses $e = 0$. After observing $w1, w2$, both types choose the highest offer if $w1, w2$ are different, and firm 1 if they are the same.
  - *Firms' belief*: Each firm believes that a worker is type H if he chooses $e = e^*$, and type L otherwise.
  - *Firms' strategies*: Each firm offers the wage H to a worker who chooses $e = e^*$, and L to a worker who chooses any other value of e.

Finding WSE

- Let's check that the conditions for consistency of beliefs and optimality of strategies are satisfied.
  - Consistency of beliefs: take the worker's strategy as given.
  - The only information sets of the firm that are reached with positive probability are after $e = 0$ and $e = e^*$; at all the rest, the firms' beliefs may be anything.
  - At the information set after $e = 0$, the only correct belief is $P(H|e = 0) = 0$.
  - At the information set after $e = e^*$, the only correct belief is $P(H|e = e^*) = 1$. So these beliefs are consistent.
  - Optimality of firm's strategy: Each firm's payoff is 0, given its beliefs and strategy.
  - If a firm deviates by offering a higher wage, it will make a negative profit.
  - If it deviates by offering a lower wage, it gets a payoff of 0 since the worker will choose the other firm. So, there is no incentive to deviate.

Finding WSE

- Optimality of worker's strategy: In the last subgame, the worker's strategy of choosing the higher wage is clearly optimal. Let's consider the worker's choice of e:
  - Type H: If the worker maintains the strategy and chooses $e = e^*$, he will get a wage offer of H and his payoff will be $H - \frac{e^*}{H}$.
  - If the worker deviates and chooses any other e, he will get a wage offer of L and his payoff will be $L - \frac{e}{H}$.
  - The highest possible payoff when deviating is when $e = 0$, which gives a payoff of L.
  - Therefore, in order for our hypothetical equilibrium to be optimal, we need $H - \frac{e^*}{H} \geq L$, or

$$e^* \leq H(H - L)$$

Finding WSE

- Type L: If the worker maintains the strategy and chooses $e = 0$, he will get a wage offer of L and his payoff will be L.
- If the worker deviates and chooses anything but e^*, he still gets a wage offer of L and a lower payoff of $L - \frac{e}{L}$.
- If the worker deviates and chooses e^* (i.e. imitates a H-type) then he gets a wage offer of H, for a total payoff of $H - \frac{e^*}{L}$.
- For our hypothetical equilibrium to be optimal, we need $L \geq H - \frac{e^*}{L}$ or

$$e^* \geq L(H - L)$$

Conditions for Equilibria

- Combining these requirements, the condition for this equilibrium to be optimal is:
$$L(H-L) \leq e^* \leq H(H-L)$$
- If this is satisfied, then separating equilibria exist in which H-type workers can be distinguished from L-type workers by their choice of e.
- This is not the only type of equilibrium that exists: there may also exist pooling equilibria, given the same values of H and L, in which both types of workers choose the same amount of education.

	C	D
C	2,2	0,3
D	3,0	1,1

- Let's recall the Prisoner's Dilemma.
- Here, we are labeling the actions for each player as *Cooperate* or *Defect*.
- As we have seen, this game has a single Nash equilibrium (D, D), where both players choose *Defect*.
- In real life, however, people frequently manage to sustain cooperation, in contrast to the theoretical prediction of the Prisoner's Dilemma model.

- One possible explanation for this is that playing Prisoner's Dilemma only once misses a key feature of the real world: that agents interact *repeatedly*.
- If agents know they will interact again in the future, defecting in one time period may be punished by reciprocal defecting in the future.
- Agents can develop a *reputation* for cooperating or defecting.
- We will study a specific case of repeated interaction, where the same agents meet in several periods and play the Prisoner's Dilemma in each period.
- In order to analyze this situation, we will need to define what a strategy is, and what preferences are for games played in several periods.

Announcements

- Homework #4 is due next week.