CUR 412: Game Theory and its Applications, Lecture 2

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Announcements

- Homework #1 will be posted on the course website today.
- Due on 3/22.
- The numbers of the exercises **may be different** in the electronic versions of the textbook! Please **check the name** of the exercise is the same.

Review of Lecture 1

- A *strategic game* is a model of a situation with many interacting decision-makers.
- A game has three parts:
  1. **Players** (the decision-makers)
  2. For each player, a set of **actions**. An *action profile* is a list of everyone's chosen action
  3. For each player, **preferences** over the set of action profiles (usually represented by a *payoff function*).

The Prisoner's Dilemma

- Players: two suspects to a crime, held by the police
- Actions: each suspect can choose to be *Quiet*, or *Fink* (inform on the other suspect)
- Preferences:
  - Suspect 1: $(F, Q) > (Q, Q) > (F, F) > (Q, F)$
  - Suspect 2: $(Q, F) > (Q, Q) > (F, F) > (F, Q)$
- These preferences can be represented by *payoff functions*:
  - Suspect 1: $u_1(F, Q) = 3, u_1(Q, Q) = 2, u_1(F, F) = 1, u_1(Q, F) = 0$
  - Suspect 2: $u_2(F, Q) = 0, u_2(Q, Q) = 2, u_2(F, F) = 1, u_2(Q, F) = 3$

Bi-Matrix Form of Prisoner's Dilemma

		Player 2	
		Q	F
Player 1	Q	2,2	0,3
	F	3,0	1,1

Nash Equilibrium

- This solution concept assumes that:
  - Players are rational (i.e. choose the highest payoff), given beliefs about other players
  - Beliefs of all players are correct
- We want to find an outcome that is a *steady state*, that is, starting from that outcome, no player wants to deviate.
- If an action profile a^* is a steady state, then all the players must *not* have other actions that they could play, that are *more preferable* to their current action in a^*.
- **Definition**: The action profile a^* in a strategic game is a **Nash Equilibrium** if, for every player i and every action b_i of player i, a^* is *at least* as preferable for player i as the action profile (b_i, a^*_{-i}):

$$u_i(a^*) \geq u_i(b_i, a^*_{-i}) \quad \text{for every action } b_i \text{ of player } i$$

Prisoner's Dilemma

		Player 2	
		Q	F
Player 1	Q	2,2	0,3
	F	3,0	1,1

- (F, F) is the unique Nash equilibrium.

BoS

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

- Two Nash equilibria: $(Bach, Bach)$ and $(Stravinsky, Stravinsky)$.

Matching Pennies

	Head	*Tail*
Head	1,-1	-1,1
Tail	-1,1	1,-1

- There is *no* Nash equilibrium.

Nash Equilibrium

Player 2

Player 1		Q	F
	Q	2000,2000	0,2001
	F	2001,0	1,1

- Note that Nash Equilibrium is not the "best possible" outcome, or the outcome which maximizes everyone's payoffs.
- It is a steady state under the condition that each player does not want to deviate *unilaterally*.
- We can imagine other solution concepts that allow multiple players to work together.
- NE also says nothing about how hard it is for players to discover which outcomes are steady states.

Guess $\frac{2}{3}$ of the Average

- Let's look at an example of a game with a *continuous* action space.
- There are n players. Each player chooses a real number between 0 and 100.
- The player who chooses the number that is closest to $\frac{2}{3}$ of the average of all the numbers wins, and gets a payoff of 1.
- If there is a tie between k players, then each winner gets a payoff of $\frac{1}{k}$.
- All other players get a payoff of 0.

Guess $\frac{2}{3}$ of the Average

- The outcome where all players choose 0 is a Nash equilibrium, since a player who deviates will get a lower payoff of 0.
- It turns out that this is also the unique Nash equilibrium.
- To prove this, we must show that with any other set of numbers, at least one player has an incentive to deviate.

Strict versus Weak Nash Equilibrium

- In a Nash equilibrium, each player's equilibrium action has to be *at least as good* as every other action, not necessarily better.
- Consider the following game:

	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0

- (T, L) is the unique Nash equilibrium.
- However, when Player 2 plays L, Player 1 is indifferent between T and B.
- This is called a *non-strict* or *weak* Nash equilibrium.

Strict versus Weak Nash Equilibrium

- **Definition**: The action profile a^* in a strategic game is a **strict Nash equilibrium** if, for every player i and every action $b_i \neq a_i^*$ of player i, a^* is *strictly* preferred by player i to the action profile (b_i, a_{-i}^*):

$$u_i(a^*) > u_i(b_i, a_{-i}^*) \quad \text{for every action } b_i \neq a_i^* \text{ of player } i$$

- A weak Nash equilibrium is in some sense, less "reasonable" than a strict Nash equilibrium, since a tiny change in payoffs could cause it to not be a Nash equilibrium.

Best Response Functions

- Suppose that the players *other* than Player i play the action list a_{-i}.
- Let $B_i(a_{-i})$ be the set of Player i's best (i.e. payoff - maximizing) actions, given that the other players play a_{-i}. (There may be more than one).
- B_i is called the **best response function** of Player i.
- B_i is a *set-valued* function, that is, it may give a result with more than one element.
- Every member of $B_i(a_{-i})$ is a **best response** of Player i to a_{-i}.

Using Best Response Functions to find Nash Eq.

- **Proposition**: The action profile a^* is a Nash equilibrium if and only if every player's action is a best response to the other players' actions:

$$a_i^* \in B_i(a_{-i}^*) \quad \text{for every player } i \tag{1}$$

- If the best-response function is *single-valued*:
  - Let $b_i(a_i^*)$ be the *single member* of $B_i(a_{-i}^*)$, i.e. $B_i(a_{-i}^*) = \{b_i(a_i^*)\}$. Then condition 1 is equivalent to:

$$a_i^* = b_i(a_{-i}^*) \quad \text{for every player } i \tag{2}$$

- If the best-response function is single-valued and there are 2 players, condition 1 is equivalent to:

$$a_1^* = b_1(a_2^*)$$

$$a_2^* = b_2(a_1^*)$$

Prisoner's Dilemma

	Q	F
Q	2,2	0,3
F	3,0	$\underline{1},\overline{1}$

- $B_i(Q) = \{F\}$ for $i = 1, 2$
- $B_i(F) = \{F\}$ for $i = 1, 2$
- At (F, F), Player 1 is playing one of his best responses, F, to Player 2's action, F.
- At the same time, Player 2 is playing one of *his* best responses, F, to Player 1's action, F.

BoS

	Bach	*Stravinsky*
Bach	$\underline{2}, \overline{1}$	0, 0
Stravinsky	0, 0	$\underline{1}, \overline{2}$

- $B_i(Bach) = \{Bach\}$ for $i = 1, 2$
- $B_i(Stravinsky) = \{Stravinsky\}$ for $i = 1, 2$

Matching Pennies

	Head	*Tail*
Head	$\underline{1}$,-1	-1,$\overline{1}$
Tail	-1,$\overline{1}$	$\underline{1}$,-1

- $B_1(Head) = \{Head\}$
- $B_2(Head) = \{Tail\}$
- $B_1(Tail) = \{Tail\}$
- $B_2(Tail) = \{Head\}$

	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0

- $B_1(L) = \{T, B\}$
- $B_1(M) = \{T\}$
- $B_1(R) = \{B\}$
- $B_2(T) = \{L, R\}$
- $B_2(B) = \{M\}$

Finding Nash equilibrium with Best-Response functions

- We can use this to find Nash equilibria when the action space is continuous.
- Step 1: Calculate the best-response functions.
- Step 2: Find an action profile a^* that satisfies:

$$a_i^* \in B_i(a_{-i}^*) \quad \text{for every player } i$$

- Or, if every player's best-response function is single-valued, find a solution of the n equations (n is the number of players):

$$a_i^* = b_i(a_{-i}^*) \quad \text{for every player } i$$

Example: synergistic relationship (37.2 in book)

- Two individuals.
- Each decides how much effort to devote to relationship.
- Amount of effort a_i is a non-negative real number (so the action space is infinite)
- Payoff to Player i: $u_i(a_i) = a_i \cdot (c + a_j - a_i)$, where $c > 0$ is a constant.

Finding the Nash Equilibrium

- Construct players' best-response functions:
- Player i's payoff function: $u_i(a_i) = a_i \cdot (c + a_j - a_i)$
- Given a_j, this becomes a quadratic: $u_i(a_i) = a_i \cdot c + a_i \cdot a_j - a_i^2$
- Best response to a_j is to choose a_i that *maximizes* this function.
- Since it is *concave*, we can use calculus to find the maximizer.
- Take the derivative and set to 0.

$$\frac{\partial u_i}{\partial a_i} = c + a_j - 2a_i = 0$$

$$\rightarrow a_i = \frac{c + a_j}{2}$$

- So, best response functions are:
  - $b_1(a_2) = \frac{c+a_2}{2}$
  - $b_2(a_1) = \frac{c+a_1}{2}$

Finding the Nash Equilibrium

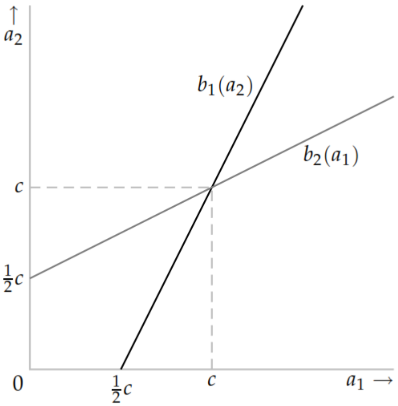
- The pair (a_1, a_2) is a Nash equilibrium if $a_1 = b_1(a_2)$ and $a_2 = b_2(a_1)$.
- Solving the two equations

$$a_1 = \frac{c + a_2}{2}$$

$$a_2 = \frac{c + a_1}{2}$$

- gives a unique solution (c, c).
- Therefore, this game has a unique Nash equilibrium: $a_1 = c, a_2 = c$.

Finding the Nash Equilibrium



- The intersection of $b_1(a_2) = \frac{c+a_2}{2}$ and $b_2(a_1) = \frac{c+a_1}{2}$ is the Nash equilibrium.
- Note that using calculus to find the best response requires that the payoffs are *concave*.

Strictly Dominated Actions

- A player's action is *strictly dominated* by another action if it gives a lower payoff, regardless of what other players do.
- **Definition**: Player i's action b_i **strictly dominates** action b_i' if

$$u_i(b_i, a_{-i}) > u_i(b_i', a_{-i}) \quad \text{for every } a_{-i}$$

- We say that an action is *strictly dominated* if some other action exists that strictly dominates it.
- A strictly dominated action cannot be a best response to any actions of the other players, because some other action exists that gives a higher payoff.
- In fact, a rational player will never play a strictly dominated action, regardless of beliefs about other players.
- So, when looking for Nash equilibria, outcomes with strictly dominated actions are eliminated.

Strictly Dominated Actions

	L	C	R
T	4,2	3,0	1,1
M	1,2	2,4	0,3
B	1,1	4,2	2,4

- For Player 1, M is a strictly dominated action.
- Therefore, a rational player will never play M, and we can eliminate it when looking for NE.
- Once we eliminate M, then C becomes strictly dominated.
- We can keep going until no more actions can be eliminated.
- This is called *iterated elimination* of strictly dominated strategies.
- It can be proven that NE (if any exist) will survive iterated elimination.

Strictly Dominant Actions

- A *strictly dominant* action is one that strictly dominates every other action (such an action may not exist).
- If a rational player has a strictly dominant action, he will play it, because it gives the highest payoff, regardless of what other players do.
- Recall that the Nash Equilibrium solution concept makes two assumptions:
  - Players are rational (i.e. payoff-maximizing), given their beliefs about other players
  - Beliefs are correct
- A different solution concept is the *dominant strategy equilibrium*, that only assumes players are rational.
- An action profile is a dominant strategy equilibrium if all actions played are dominant strategies.
- If all players have a dominant strategy, they will play it and that is the dominant strategy equilibrium.

Prisoner's Dilemma

	Q	F
Q	2,2	0,3
F	3,0	1,1

- For both players, F strictly dominates Q: regardless of the other player's action, F gives a higher payoff.
- This eliminates all outcomes where Q is played as Nash equilibria.

BoS

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

- Neither action is strictly dominated.
- dominant strategy equilibria are subset of Nash equilibria.

Illustration: Voting (2.9.3)

- Suppose there is an election with two candidates, A and B.
- There are an *odd* number of voters, so there are no ties.
- Game definition:
  - Players: n citizens, n is odd.
  - Actions: Each citizen can choose to vote for A or B.
  - Preferences: Each voter has a preferred candidate; the voter prefers the outcome in which his candidate wins, but is otherwise indifferent (e.g. to his own vote, or to the total number of votes). Assume a majority prefer candidate A.

Illustration: Voting (2.9.3)

- If everyone votes for their preferred candidate, that is a NE (but not the only one).
- Voting for your *less preferred* candidate is weakly dominated by voting for your *preferred* candidate.
- In some situations, switching to your *preferred* candidate will increase payoff (if you are a pivotal voter)
- In other situations, it has no effect on payoff (if one vote cannot change the outcome)
- But it can never decrease payoff. Therefore, it is weakly dominated
- The conditions for Nash Equilibrium depends on each *individual*'s incentives to deviate.
- Easy to construct situations with a "bad" outcome, if it requires more than one agent to affect outcome

Illustration: Voting (2.9.3)

- Assume there are 5 players, 3 prefer A, 2 prefer B.
- Suppose each player votes for their preferred candidate (3 A, 2 B, A wins). This is a weak NE.
- Suppose all players vote for A. This is a weak NE.
- Suppose all players vote for B. This is also a weak NE, since no single player can change the winner by acting alone.
- This situation would not occur if, e.g. voters got a tiny payoff from voting for their preferred candidate (even if he does not win).
- Therefore, we may want to consider weak NE "less reasonable".

Equilibrium in a Single Population

- A Nash equilibrium corresponds to a steady state of interaction between several populations, one for each player in the game.
- What if members of a *single* population interact?
- All players have same actions and payoffs.
- For two players, a game is *symmetric* if:
  - Each player has the same set of actions
  - Payoffs depend only on players' actions, not whether the player is player 1 or 2.
- **Definition**: A two-player strategic game is **symmetric** if the player's sets of actions are the same and

$$u_1(a_1, a_2) = u_2(a_2, a_1) \quad \text{for all } (a_1, a_2)$$

Symmetric 2x2 Games

	A	B
A	w,w	x,y
B	y,x	z,z

- Two-player symmetric games with two actions have this form.
- Prisoner's Dilemma is symmetric.
- Bach vs. Stravinsky, Matching Pennies are not symmetric.

Symmetric Nash Equilibrium

- Suppose we want to model a steady state of a situation where players come from a single population.
- There is only one role in the game, so steady state is a *single* action used by every participant.
- An action a^* is a steady state if (a^*, a^*) is a Nash equilibrium.
- **Definition**: A pair of actions (a_1^*, a_2^*) in a symmetric two-player game is a **symmetric Nash equilibrium** if it is a Nash equilibrium and $a_1^* = a_2^*$.

Symmetric Nash Equilibrium: Examples

	Left	*Right*
Left	1,1	0,0
Right	0,0	1,1

- Approaching Pedestrians: two people are walking towards each other. Each is better off when they step in the same direction, avoiding a collision.
- There are two symmetric Nash equilibria: $(Left, Left)$ and $(Right, Right)$.

	X	Y
X	0,0	1,1
Y	1,1	0,0

- This game is symmetric, but has no symmetric Nash equilibria.
- (X, Y) and (Y, X) are Nash equilibria, but do not satisfy the condition $a_1 = a_2$.

Next Week

- Please read the rest of Chapter 2 and Chapter 3.1-3.3.
- Homework #1 will be posted on the course website today.
- Due on 3/22.
- The numbers of the exercises **may be different** in the electronic versions of the textbook! Please **check the name** of the exercise is the same.