CUR 412: Game Theory and its Applications, Lecture 3

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March 15, 2015

Homework #1

- Homework #1 is due next week.
- The numbers of the exercises **may be different** in the electronic versions of the textbook! Please **check the name** of the exercise is the same.

Review of Last Week

- *Best Response Function*: Suppose the action list for players other than i is a_{-i}. The *best response* function of player i, denoted $B_i(a_{-i})$, is the set of actions of player i that give the highest payoff (out of all possible actions).
- Nash equilibrium is equivalent to the condition that each player is playing a best response to the other players' actions.
- We can use this to find the Nash equilibria of games, by finding the intersection of best response functions.
- This is always possible for games with a finite number of actions (i.e. bi-matrix games).
- It may be possible in games with an infinite action space (e.g. if the payoff function is concave).

Review of Last Week

- *Strict Dominance*: Action A *strictly dominates* action B if A gives a strictly higher payoff, no matter what actions the other players play.
- An action is *strictly dominated* if there is some other action that strictly dominates it.
- A strictly dominated action will never be played in a Nash equilibrium (in fact, it is never rational to play such an action).
- We can eliminate strictly dominated actions when searching for NE.

Review of Last Week

- A *symmetric* 2-player, 2-action game is one where the roles of Player 1 and Player 2 are interchangeable.
- We use this to model a situation where players are drawn from a single population.

Weak Domination

- A player's action is *weakly dominated* if the player has another action that is *never worse*, and *better in at least one case*, depending on the other players' actions.
- **Definition**: Player i's action b_i **weakly dominates** action b_i' if

$$u_i(b_i, a_{-i}) \geq u_i(b_i', a_{-i}) \quad \text{for every } a_{-i}$$

- with at least one strict inequality for some list a_{-i} of the other players' actions:

$$u_i(b_i, a_{-i}) > u_i(b_i', a_{-i}) \quad \text{for at least one } a_{-i}$$

Weak Domination: Example

	L	R
T	1	0
M	2	0
B	2	1

- Only player 1's payoffs are shown in this matrix.
- For player 1, M weakly dominates T, and B weakly dominates M.
- B strictly dominates T.

Weakly Dominated Actions

- Can a weakly dominated action be played in a Nash equilibrium?

	B	C
B	1,1	0,0
C	0,0	0,0

- In this example, B weakly (but not strictly) dominates C.
- Both (B, B) and (C, C) are Nash equilibria, but only (B, B) is a strict Nash equilibrium.
- A weakly dominated action may be played in a Nash equilibrium; this might be considered "unreasonable" (recall the election example).
- Recall: in a *strict* Nash equilibrium, each player's equilbrium action gives a strictly higher payoff, given the other players' actions.
- Therefore, in a strict NE, no equilibrium action is weakly dominated.

Illustration: Collective Decision-Making (2.9.4)

- Suppose a group of people are deciding on a policy that affects everyone. Assume that the policy can be represented by a single number.
- For example: tax level, location of a park, "left" vs. "right"...
- Each person will announce a policy, and the *median* of the announcements will be chosen.
  - Players: n citizens, n is odd. Each citizen i has a favorite policy, a number x_i^*.
  - Actions: Each citizen chooses to announce a real number x_i.
  - Preferences: For citizen i, his payoff is the *negative* of the distance from his favorite policy x_i^*, to the *median* of everyone's announced policies.

Properties of the Median

- The *median* of N numbers is a number that half of the numbers are below, and half are above.
- If N is odd, then the median is the middle number. For example: the median of $3, 3, 5, 9, 11$ is 5.
- If N is even, then the median is the average of the middle two numbers. For example: the median of $3, 3, 5, 6, 9, 11$ is 5.5.
- If N is odd, if the numbers above the median increase, the value of the median is not affected (likewise, if the numbers below the median decrease).
- This is in contrast to the *average* of N numbers, which is affected by a change to any of the numbers.
- For example, when comparing the wealth of "ordinary" citizens across countries, the median of wealth (instead of average wealth) is frequently used, since it is less affected by a few very rich people.

Let's Play Collective Decision-Making

- I will choose 5 people, and we will assign each person a preferred policy from 1 to 5.
- Each person will choose a number x_i.
- The chosen policy will be the median of $x_1, ..., x_5$.
- Player i's payoff will be the negative of the distance from x_i to the median.

Illustration: Collective Decision-Making (2.9.4)

- Claim: for each player i, the action of announcing x_i^*, (i.e. being truthful) weakly dominates all other actions.
- To prove this, need to show that switching your announced x_i from x_i^* to *anything else* will never increase your payoff (but may leave it unchanged).
- In the following cases, assume that player i has chosen to announce $x_i = x_i^*$.
- Case 1: The median is equal to player i's preferred position, x_i^*. Clearly, player i cannot increase his payoff further.
- Case 2: The median is to the left (or below) x_i^*.
  - Player i would like to move the median to the right, but cannot do so (by the properties of the median).
  - Player i *can* move the median to the left, if he lowers his announced x_i far enough, but this *lowers* his payoff.
- Case 3: The median is to the right (or above) x_i^*. Same as Case 2.

Models of Oligopoly

- In microeconomics, you studied industries with many small firms, or one firm.
- Perfect competition: firms have no market power, are price takers. Result: $P = MC$
- Monopoly: one firm that faces market demand. Result: $P > MC$, consumer surplus is reduced
- With game theory, we can study an industry with more than one firm.
- Firms compete with other on price, output, quality, etc.
- Each firm's action affects the profitability of other firms.

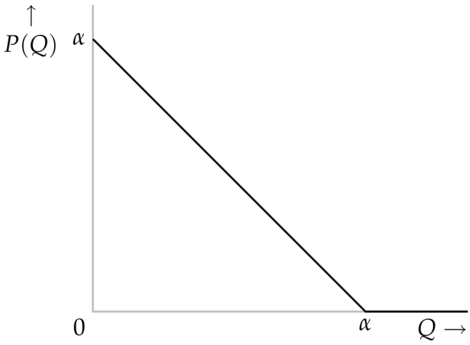
Cournot's Model of Oligopoly

- Cournot (1836) proposed that firms compete on output.
- There are n firms, producing a single good.
- The cost to firm i of producing q_i units of output is $C_i(q_i)$, where C_i is non-negative and increasing.
- All output on the market is sold at a single price, which is determined by the demand curve for the good.
- Let Q be the total output of all firms: $Q = q_1 + q_2 + \dots q_n$
- Market demand is specified by an *inverse demand* function $P(Q)$, which gives the market price as a function of total output.

Cournot's Model of Oligopoly

- We assume each firm has market power, and all firms are aware of their own (and all other firms') market power.
- Firm i's revenue is $q_i \cdot P(q_1 + ... + q_n)$
- Profit is $\pi_i(q_1, ...q_n) = q_i \cdot P(q_1 + ... + q_n) - C_i(q_i)$
- Cournot's Oligopoly Game:
  - Players: the n firms
  - Actions: each firm chooses its output q_i, a non-negative number
  - Preferences: Payoff of an outcome $(q_1, ...q_n)$ is given by profit to the firm

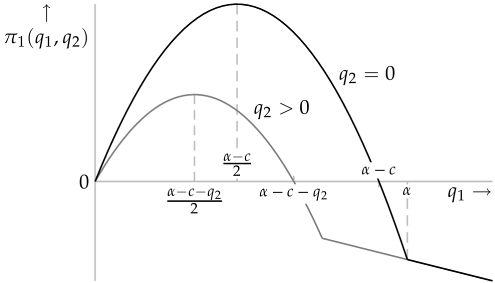
Cournot Oligopoly with Two Firms



- Assume $n = 2$.
- Constant unit cost: $C_i(q_i) = c \cdot q_i$, where $c < \alpha$
- Inverse demand function:

$$P(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$$

Payoff Function



- Firm 1's profit is: $\pi_1(q_1, q_2) = q_1 \cdot (P(q_1 + q_2) - c)$

$$= \begin{cases} q_1 \cdot (\alpha - c - q_2 - q_1) & \text{if } q_1 \leq \alpha - q_2 \\ -c \cdot q_1 & \text{if } q_1 > \alpha - q_2 \end{cases}$$

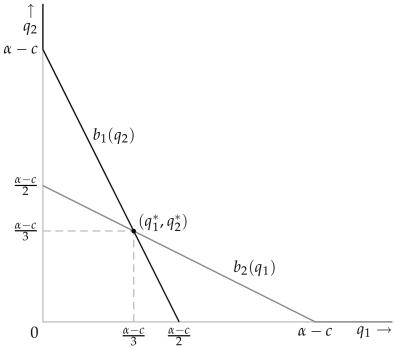
Best Response Function

- The maximum of the quadratic is at $q_i = \frac{1}{2}(\alpha - c - q_2)$
- If the quadratic region is above 0 (positive profit), this is the best response, from $q2 = 0$ up to $q_2 = \alpha - c$
- If it's below 0, then the firm is making a loss at any output level; best response is to produce nothing, $q_i = 0$
- Best response function:

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c - q_2) & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c \end{cases}$$

- Firm 2's best response function is symmetric, same as Firm 1's with q_1, q_2 reversed

Nash Equilibrium



- Condition for Nash equilibrium:

$$q_1^* = b_1(q_2^*) \quad \text{and} \quad q_2^* = b_2(q_1^*)$$

$$\rightarrow q_1 = \frac{1}{2}(\alpha - c - q_2), q_2 = \frac{1}{2}(\alpha - c - q_1)$$

- Solution: $q_1^* = q_2^* = (\alpha - c)/3$

Properties of Cournot Equilibrium

- Each firm's profit at equilibrium is $(\alpha - c)^2/9$.
- Total output is $\frac{2}{3}(\alpha - c)$
- Under perfect competition, $P = c \rightarrow Q = \alpha - c$
- Under monopoly, $Q = \frac{1}{2}(\alpha - c)$

Bertrand's Model of Oligopoly

- In Cournot's model, firms all charged the same price, and competed on output.
- Bertrand (1883) criticized Cournot's formulation, arguing that firms compete on price instead.
- Assumptions:
  - n firms producing a single good
  - Cost to firm i of producing q_i units of output: $C_i(q_i)$, where C_i is non-negative and increasing
  - Consumers *only* buy from firms with the lowest price. If many firms offer the same lowest price, they split demand equally.
  - Demand function $D(p)$
  - Firms produce what is demanded.

Bertrand's Oligopoly Game

- Assume $n = 2$.
- Firm 1's profit:

$$\pi_1(p_1, p_2) = \begin{cases} p_1 D(p_1) - C_1(D(p_1)) & \text{if } p_1 < p_2 \\ \frac{1}{2} p_1 D(p_1) - C_1(\frac{1}{2} D(p_1)) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

- Players: 2 firms
- Actions: Each firm chooses its price, a non-negative number
- Preferences: Payoff of an outcome $(p_1, ...p_n)$ is given by profit to the firm

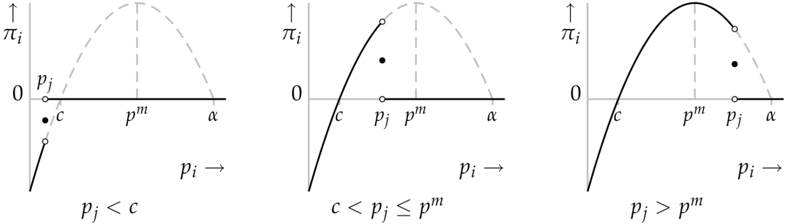
Bertrand's Oligopoly Game

- Assume constant unit cost: $C_i(q_i) = c \cdot q_i$
- Demand function: $D(p) = \alpha - p$ for $p \leq \alpha$, $D(p) = 0$ for $p > \alpha$
- Assume $c < \alpha$.
- Firm i has constant unit cost c, so will make unit profit $p_i - c$.
- Firm i's profit:

$$\pi_1(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Payoff Function

$$\pi_1(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$



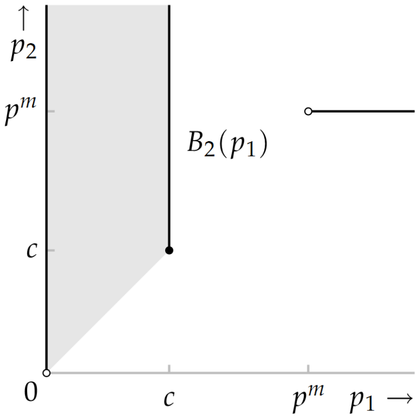
Best Response Function

- Let p^m be the price that maximizes the quadratic portion of the payoff curve.
- Case 1: $p_2 < c$.
  - Firm 2 is losing money.
  - Any price above p_2 is a best response.
  - Firm 1 will attract no customers and make zero profit.
- Case 2: $p_2 = c$.
  - Firm 2 is setting $P = MC$ and making zero profit.
  - Any price $\geq c$ is a best response.
  - If firm 1 chooses $p_1 = c$, splits demand with firm 2 and makes zero profit.
  - If firm 1 chooses $p_1 > c$, attracts no customers and makes zero profit.

Best Response Function

- Case 3: $c < p_2 < p^m$.
  - Firm 2 is charging above marginal cost.
  - Firm 1 can take all the customers while still making a profit by charging a lower price.
  - There is no best response; profit can always be increased (by smaller and smaller amounts) by charging a higher price that is still less than p_2.
- Case 4: $p_2 > p^m$.
  - p^m is the unique best response.
  - Firm 1 will take all the customers and maximize positive profits.

Best Response Function



- This shows Player 2's best response function.
- Player 1's best response function is identical, with p_1 and p_2 reversed.

Nash Equilibrium

- Let's consider the possible cases of (p_1, p_2) to find Nash equilibria. Recall that at NE, no player has an incentive to deviate.
- So, if starting from (p_1, p_2), any player has a better response, it is not a NE.
- Case 1: $p_1 = p_2 = c$
  - Playing c is a best response to the other player playing c. So, it is a NE.
- Case 2: $p_i < c$ for either player
  - A firm playing $p_i < c$ is making negative profit, so a better outcome is to play $p_i = c$ and make zero profit. Not a NE.
- Case 3: $p_i = c, p_j > c$
  - Player i can increase his profit by raising p_i to anything less than p_j. Not a NE.

Nash Equilibrium

- Case 4: $p_i > c, p_j > c$
  - Suppose $p_i \geq p_j$. Player i can undercut player j by lowering p_i to a price just below p_j, take all the customers, and make a positive profit. Not a NE.

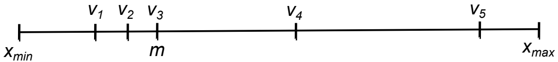
Bertrand Paradox

- Nash equilibrium outcome is the same as perfect competition: $P = MC$, firms make zero profits
- Intuitively, it should require many firms to drive down price to marginal cost.
- However, in Bertrand's model, only two firms are required.
- This is due to the assumption that customers only buy from the lowest price.
- Also, if firms could collusively fix prices, they could behave like a monopolist.

Hotelling's Model of Electoral Competition

- This is a widely used model in political science and industrial organization, Hotelling's "linear city" model.
- Players choose a location on a line; payoffs are determined by how much of the line is closer to them than other players.
- Here, location represents a position on a *one-dimensional* political spectrum, but it can also represent physical space or product space.

Location on Political Spectrum

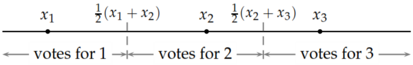


- Political position is measured by position on an interval of numbers
- x_{min} is the most "left-wing" position, x_{max} is the most "right-wing" position
- Voters are located at fixed positions somewhere on the line. This position represents their "favorite position".
- In this example, there are five voters with favorite positions at $v_1 \ldots v_5$.
- The *median* position m is the position such that half of voters are to the left or equal to m, and the other half are to the right or equal to m.
- Voters dislike positions that are farther away from them on the line. They are indifferent between positions to their left and right that have the same distance.

Attracting Voters Based on Position

- Candidates can choose their position.
- Assume that voters vote for a candidate based only on distance to the voter's position. They always vote for the closest candidate.
- If there is a tie (two candidates with the same distance), the candidates will split the vote.
- Therefore, each candidate will attract all voters who are closer to him than any other candidate.

Attracting Voters Based on Position



- Suppose there are three candidates who choose positions at x_1, x_2, x_3.
- All voters to the left of x_1 will vote for x_1. Likewise, all voters to the right of x_3 will vote for x_3.
- Between candidates x_1 and x_2, each candidate will attract voters up to the midpoint $(x_1 + x_2)/2$.
- The candidate that attracts the most votes wins. Ties are possible.
- Candidates' most preferred outcome is to win. A tie is less preferable; the more the tie is split, the less preferred.
- Losing is the least preferable outcome.

Candidates' Payoff Function

- Payoffs can be represented by this function:

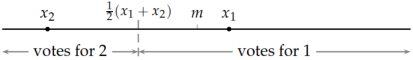
$$u_i(x_1, ...x_n) = \begin{cases} n & \text{if candidate wins} \\ k & \text{if candidate ties with } n-k \text{ other candidates} \\ 0 & \text{if candidate loses} \end{cases}$$

- Definition of Hotelling's Game of Electoral Competition:
  - Players: the candidates
  - Actions: each candidate can choose a position (a number) on the line
  - Preferences: Each candidate's payoff is given by the function above.

Let's Play the 2-Person Game

- Assume voters (or consumers) are uniformly distributed along a line that begins at 0 and ends at 100.
- I will choose 2 players.
- Each player will write down a number from 0 to 100. This is their position on the line.
- Calculate the portion of the line that is closer to each player.
- The player with the larger portion wins (if portions are equal, there is a tie).

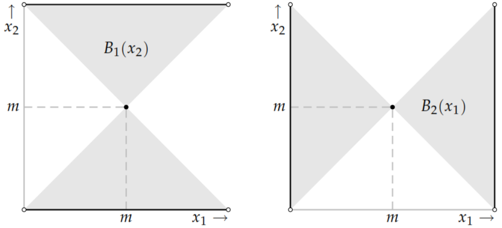
Two Candidates



- Suppose there are two candidates that choose positions x_1, x_2.
- The median position (half of voters are on the left, half on the right) is m.
- Let's examine the best response function of player 1 to x_2.

- Case 1: $x_2 < m$
  - Player 1 wins if $x_1 > x_2$ and $(x_1 + x_2)/2 < m$. Every position between x_j and $2m - x_j$ is a best response.
- Case 2: $x_2 > m$
  - By the same reasoning, every position between $2m - x_j$ and x_j is a best response.
- Case 3: $x_2 = m$
  - Choosing m results in a tie; any other choice results in a loss. Therefore, $x_1 = m$ is the best response.

Best Response Function



- Best response function is:

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x1 < 2m - x2\} & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ \{x_1 : 2m - x_2 < x1 < x2\} & \text{if } x_2 > m \end{cases}$$

- Unique Nash equilibrium is when both candidates choose m.

Direct Argument for Nash Equilibrium

- At (m, m), any deviation results in a loss.
- At any other position:
  - If one candidate loses, he can get a better payoff by switching to m.
  - If there is a tie, either candidate can get a better payoff by switching to m.

Implications of Equilibrium

- Conclusion: competition between candidates drives them to take similar positions at the median favorite position of voters
- In physical or product space: competing firms are driven to locate at the same position, or offer similar products
- This is known as "Hotelling's Law" or "principle of minimum differentiation"
- Requires the one-dimensional assumption on voter/consumer preferences.
- If there is more than one dimension (e.g. consumers care about both price and quality), this result may not hold

Next Week

- Please read the rest of Chapter 3 and 4.1-4.6 in Chapter 4.
- Email me if you do not have the book.
- Homework #1 is posted on the course website
- Due next week.
- The numbers of the exercises **may be different** in the electronic versions of the textbook! Please **check the name** of the exercise is the same.