CUR 412: Game Theory and its Applications, Lecture 4

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March 22, 2015

Homework #1

- Homework #1 will be due at the end of class today.
- Please check the website later today for the solutions to HW1 and HW2, which is due on 4/5.

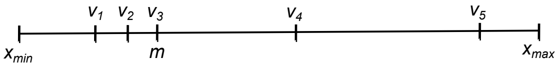
Review of Last Lecture

- Last week, we looked at a classic application of game theory in economics: the behavior of oligopolies.
- In Cournot oligopoly, firms choose their quantity of output.
- Nash equilibrium outcome: firms split the market. Output and profits are in between the case of monopoly and perfect competition.
- In Bertrand oligopoly, firms choose their price. Consumers only buy from the lowest price seller.
- Nash equilibrium outcome: firms set $P = MC$, make zero profit. Same as perfect competition.

Hotelling's Model of Electoral Competition

- This is a widely used model in political science and industrial organization, Hotelling's "linear city" model.
- Players choose a location on a line; payoffs are determined by how much of the line is closer to them than other players.
- Here, location represents a position on a *one-dimensional* political spectrum, but it can also represent physical space or product space.

Location on Political Spectrum

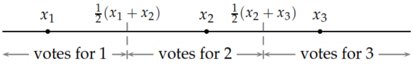


- Political position is measured by position on an interval of numbers
- x_{min} is the most "left-wing" position, x_{max} is the most "right-wing" position
- Voters are located at fixed positions somewhere on the line. This position represents their "favorite position".
- In this example, there are five voters with favorite positions at $v_1 \ldots v_5$.
- The *median* position m is the position such that half of voters are to the left or equal to m, and the other half are to the right or equal to m.
- Voters dislike positions that are farther away from them on the line. They are indifferent between positions to their left and right that have the same distance.

Attracting Voters Based on Position

- Candidates can choose their position.
- Assume that voters vote for a candidate based only on distance to the voter's position. They always vote for the closest candidate.
- If there is a tie (two candidates with the same distance), the candidates will split the vote.
- Therefore, each candidate will attract all voters who are closer to him than any other candidate.

Attracting Voters Based on Position



- Suppose there are three candidates who choose positions at x_1, x_2, x_3.
- All voters to the left of x_1 will vote for x_1. Likewise, all voters to the right of x_3 will vote for x_3.
- Between candidates x_1 and x_2, each candidate will attract voters up to the midpoint $(x_1 + x_2)/2$.
- The candidate that attracts the most votes wins. Ties are possible.
- Candidates' most preferred outcome is to win. A tie is less preferable; the more the tie is split, the less preferred.
- Losing is the least preferable outcome.

Candidates' Payoff Function

- Payoffs can be represented by this function:

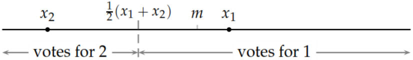
$$u_i(x_1, ...x_n) = \begin{cases} n & \text{if candidate wins} \\ k & \text{if candidate ties with } n-k \text{ other candidates} \\ 0 & \text{if candidate loses} \end{cases}$$

- Definition of Hotelling's Game of Electoral Competition:
  - Players: the candidates
  - Actions: each candidate can choose a position (a number) on the line
  - Preferences: Each candidate's payoff is given by the function above.

Let's Play the 2-Person Game

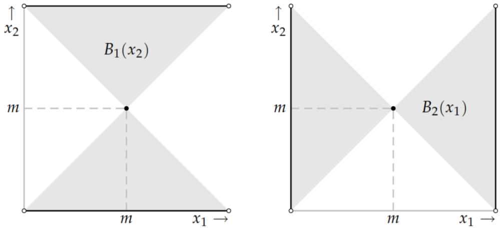
- Assume voters (or consumers) are uniformly distributed along a line that begins at 0 and ends at 100.
- I will choose 2 players.
- Each player will write down a number from 0 to 100. This is their position on the line.
- Calculate the portion of the line that is closer to each player.
- The player with the larger portion wins (if portions are equal, there is a tie).

Two Candidates



- Suppose there are two candidates that choose positions x_1, x_2.
- The median position (half of voters are on the left, half on the right) is m.
- Let's examine the best response function of player 1 to x_2.
- Case 1: $x_2 < m$
  - Player 1 wins if $x_1 > x_2$ and $(x_1 + x_2)/2 < m$. Every position between x_j and $2m - x_j$ is a best response.
- Case 2: $x_2 > m$
  - By the same reasoning, every position between $2m - x_j$ and x_j is a best response.
- Case 3: $x_2 = m$
  - Choosing m results in a tie; any other choice results in a loss. Therefore, $x_1 = m$ is the best response.

Best Response Function



- Best response function is:

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x1 < 2m - x2\} & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ \{x_1 : 2m - x_2 < x1 < x2\} & \text{if } x_2 > m \end{cases}$$

- Unique Nash equilibrium is when both candidates choose m.

Direct Argument for Nash Equilibrium

- At (m, m), any deviation results in a loss.
- At any other position:
  - If one candidate loses, he can get a better payoff by switching to m.
  - If there is a tie, either candidate can get a better payoff by switching to m.

Implications of Equilibrium

- Conclusion: competition between candidates drives them to take similar positions at the median favorite position of voters
- In physical or product space: competing firms are driven to locate at the same position, or offer similar products
- This is known as "Hotelling's Law" or "principle of minimum differentiation"
- Requires the one-dimensional assumption on voter/consumer preferences.
- If there is more than one dimension (e.g. consumers care about both price and quality), this result may not hold

War of Attrition

- This game was originally developed as a model of animal conflict.
- Two animals are fighting over prey.
- Each animal gets a payoff from getting the prey, but fighting is costly.
- Each animal chooses a time at which it will give up fighting; the first one to give up loses the prey.
- This can be applied to any kind of dispute between parties, where there is some cost to waiting.

Setup of the Game

- Two players are disputing an object. The player that concedes first loses the object to the other player.
- Time is a continuous variable that begins at 0, goes on indefinitely.
- Assume that player i places value v_i on the object (may be different from other player's value).
- If player i wins the dispute, he gains v_i in payoff.
- Time is costly. For each unit of time that passes before one side concedes, both players lose 1 in payoff.

Payoffs

- Suppose that player i concedes first at time t_i.
  - Player i's payoff: $-t_i$
  - Player j's payoff: $v_j - t_i$
- If both players concede at the same time, they split the object.
  - Player i's payoff: $v_i/2 - t_i$
  - Player j's payoff: $v_j/2 - t_i$

Let's play War of Attrition

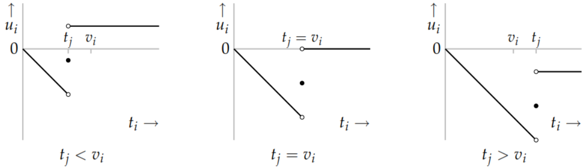
- I will choose 2 players, with valuations of 5 and 10.
- Each player will write down a number $t_i \geq 0$. Reveal them simultaneously.
- Suppose that $t_i < t_j$.
  - Player i's payoff: $-t_i$
  - Player j's payoff: $v_j - t_i$
- If $t_i = t_j$, they split the object.
  - Player i's payoff: $v_i/2 - t_i$
  - Player j's payoff: $v_j/2 - t_i$

Definition of the Game

- Players: two parties in a dispute.
- Actions: each player's set of actions is the set of concession times (a non-negative number).
- Preferences: Payoffs are given by the following function:

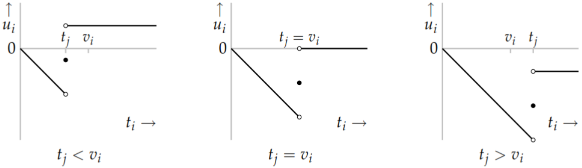
$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ v_i/2 - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j \end{cases}$$

Best Response Function



- Intuitively, if the other player is going to give up quickly, you should wait a longer time.
- But if the other player is determined to wait a long time, you should concede as soon as possible.
- Suppose player j chooses t_j.
- Case 1: $t_j < v_i$
  - Any time after t_j is a best response to t_j. Payoff: $v_i - t_j$

Best Response Function



- Case 1: $t_j < v_i$
  - Any time after t_j is a best response to t_j. Payoff: $v_i - t_j$
- Case 2: $t_j = v_i$
  - Any time after or equal to v_i is a best response. Payoff: 0
- Case 3: $t_j > v_i$
  - $t_i = 0$ is the best response. Payoff: 0

Nash Equilibrium

- (t_1, t_2) is a Nash equilibrium if and only if:
- $t_1 = 0, t_2 \geq v_1$ or
- $t_2 = 0, t_1 \geq v_2$

Implications of Equilibrium

- Players don't actually fight in equilibrium.
- Either player can concede first, even if he has the higher valuation.
- Equilibria are asymmetric: each player chooses a different action, even if they have the same value
- This can only be a stable social norm if players come from different populations (e.g. owners always concede, challengers always wait)

Let's Hold an Auction

- I will auction off 50 yuan.
- If you want to submit a bid, write down your ID number and bid (a number ≥ 0) on the piece of paper.
- I will pay the difference between 50 and the highest bid. (If it is negative, then the winner should pay me.)
- If there is a tie, the difference will be evenly divided among everyone with the highest bid.
- What do you predict the outcome will be?

Let's Hold an Auction

- This type of auction is called a *first-price, sealed-bid* auction.
  - first-price, because the winner (the highest bidder) pays his bid
  - sealed-bid, because bids are made secretly, then revealed simultaneously
- Also a type of *common-value* auction, since the value of the prize is the same to all players
- What are the Nash equilibria of this auction?

Let's Hold an Auction

- What are the Nash equilibria of this auction?
- Assume two players who bid x_1, x_2. The value of the prize is V.
- Let's find Player 1's best response to x_2.
- Suppose $x_2 < V$.
  - No matter what x_1 is, Player 1 can always improve his payoff by choosing a number closer to, and greater than, x_2.
  - No best response exists.
- Suppose $x_2 > V$.
  - Any number less than x_2 gives the same payoff of 0. This is the set of best responses.
- Suppose $x_2 = V$.
  - Any number $\leq V$ gives the same payoff of 0. This is the set of best responses.
- $x_1 = V, x_2 = V$ is the only Nash equilibrium.

Let's Hold an Auction

- Outcome is similar to Bertrand duopoly.
- Two players are enough to compete profits away to zero.

Auctions

- A good is sold to the party who submits the highest bid.
- A common form of auction:
- Potential buyers sequentially submit bids.
- Each bid must be higher than the previous one.
- When no one wants to submit a higher bid, the current highest bidder wins.
- The actual winning bid has to only be slightly higher than the second-highest bid.
- We can model this as a *second-price* auction: the winner is the highest bidder, but only has to pay the second-highest price.

Second-price, sealed-bid auction

- Assume that each person knows his valuation of the object before the auction begins, so that valuation cannot be changed by seeing how others behave.
- Therefore, there is no difference if all bids are made secretly, then simultaneously revealed: a *closed-bid* auction.
- Each player submits the maximum amount he is willing to pay.
- The highest bidder wins, and pays the second-highest price.

Setup of the Game

- There are n bidders.
- Bidder i has valuation v_i for the object. Assume that we label the bidders in decreasing order:
- $v_1 > v_2 > ... > v_n$
- Each player submits a sealed bid b_i.
- If player i's bid is the highest, he wins and has to pay the second-highest bid b_j. Payoff: $v_i - b_j$
- Otherwise, does not win. Payoff: 0
- To break ties, assume player with the highest valuation wins.

Let's Hold a Second-Price Auction

- I will choose 2 players. Each player will draw a card, which is either 5 or 8. This will be your valuation of the prize.
- We will reveal everyone's valuation.
- The players will write down their bids (a number ≥ 0) on a piece of paper.
- The highest bidder wins the prize, and pays the second-highest price.
- We will calculate everyone's payoff.

Nash Equilibrium with 2 Players

- Suppose there are 2 players with valuations v_1, v_2. Assume $v_1 > v_2$.
- Each player bids b_i.
  - If $b_1 > b_2$, Player 1's payoff is $v_1 - b_2$, Player 2's payoff is 0.
  - If $b_2 > b_1$, Player 1's 0, Player 2's payoff is $v_2 - b_1$.
  - If $b_1 = b_2$, Player 1's payoff is $v_1 - b_1$, Player 2's payoff is 0.

Best Response Function with 2 Players

- Let's find Player 1's best response function.
  - If $b_2 < v_1$, Player 1 can get a positive payoff by bidding $b_1 \geq b_2$. Best response is $b_1 \geq b_2$.
  - If $b_2 = v_1$, Player 1 will get a zero payoff with any bid. Best response is $b_1 \geq 0$.
  - If $b_2 > v_1$, Player 1 can get a zero payoff by bidding $b_1 < b_2$, or a negative payoff if $b_1 \geq b_2$. Best response is $b_1 < b_2$.
- Player 2:
  - If $b_1 < v_2$, Player 2 can get a positive payoff by bidding $b_2 > b_1$. Best response is $b_2 > b_1$.
  - If $b_1 = v_2$, Player 2 will get a zero payoff with any bid. Best response is $b_2 \geq 0$.
  - If $b_1 > v_2$, Player 2 can get a zero payoff by bidding $b_2 \leq b_1$, or a negative payoff if $b_2 > b_1$. Best response is $b_2 \leq b_1$.

Nash Equilibrium With More Than 2 Players

- If there are more than 2 players, best response function becomes very complicated.
- There are many Nash equilibria in this game. Let's examine some special cases.

All Players Bid their Valuation

- $(b_1 ... b_n) = (v_1 ... v_n)$, i.e. every player's bid is equal to his valuation.
- Player 1, who has the highest valuation, wins the object and pays v_2.
- Player 1's payoff: $v_1 - v_2 > 0$, all other players: 0
- Does anyone have an incentive to deviate?
- Player 1:
  - If Player 1 changes bid to $\geq b_2$, outcome does not change
  - If Player 1 changes bid to $< b_2$, does not win, gets lower payoff of 0
- Players 2 ... n:
  - If Player i lowers bid, still loses.
  - If Player i raises bid to above $b_1 = v_1$, wins, but gets negative payoff $v_i - v_1 < 0$

Player 1 Bids Valuation, All Others Bid 0

- $(b_1 ... b_n) = (v_1, 0...0)$, i.e. all players except player 1 bids 0
- Player 1 wins, pays 0. Payoff: v_1
- Does anyone have an incentive to deviate?
- Player 1:
  - Any change in bid results in same outcome (because of tie-breaking rules)
- Players 2 ... n:
  - If Player i raises bid to $\leq v_1$, still loses
  - If Player i raises bid to $> v_1$, wins, but gets negative payoff $v_i - v_1$
- This outcome is better off for player 1, but worse off for the seller of the object

An Equilibrium where Player 1 Doesn't Win

- $(b_1 ... b_n) = (v_2, v_1, 0, ...0)$. Player 2 wins the auction and pays price v_2. All players get a payoff of 0.
- Player 1:
  - If he raises bid to $x < v_1$, still loses.
  - If he raises bid to $x \geq v_1$ he wins, gets payoff of 0
- Player 2:
  - If he raises bid or lowers to $x > v_2$, outcome unchanged
  - If he lowers bid to $x \leq v_2$, loses auction, gets payoff of 0
- Players 3 ... n:
  - If he raises bid to $x \leq v_1$, still loses
  - If he raises bid to $> v_1$, wins, but gets negative payoff $v_i - v_1$

An Equilibrium where Player 1 Doesn't Win

- $(b_1 \ldots b_n) = (v_2, v_1, 0, \ldots 0)$. Player 2 wins the auction and pays price v_2. All players get a payoff of 0.
- Player 2's bid in this equilibrium exceeds his valuation.
- This seems risky - what if player 1 decided to bid higher than v_2?
- In a dynamic setting, player 2's bid is not credible.
- Later on, we'll study ways of showing that these kinds of outcomes are implausible.

Equilibria with Weakly Dominant Actions

- For each player i, the action v_i weakly dominates all other actions
- Player i can do no better than bidding v_i, no matter what other players bid
- If the highest bid of other players is $\geq v_i$, then:
  - If player i bids v_i, payoff is 0 (either win and pay v_i, or don't win)
  - If player i bids $b_i \neq v_i$, payoff is zero or negative
- If the highest bid of the other players is $b < v_i$, then:
  - If player i bids v_i, wins and gets payoff $v_i - b$
  - If player i bids $b_i \neq v_i$, either wins and gets same payoff, or loses and gets payoff of 0
- Second-price auction has many Nash equilibria, but the only equilibrium where each player plays a weakly dominant action is $(b_1 \dots b_n) = (v_1 \dots v_n)$.

First-price, sealed bid auction with n players

- We saw that in the first-price sealed bid auction with 2 players who had the *same* valuation, the only NE was $(b_1, b_2) = (v_1, v_2)$.
- Now, let's allow n players, and valuations can differ among players.
- Assume that if there is a tie, the winner is the player with the highest valuation.
- First, let's check the obvious case, where everyone bids their valuations: $(b_1 ... b_n) = (v_1 ... v_n)$
- The winner, Player 1, has an incentive to deviate: he can increase his payoff by lowering his bid from v_1 to v_2.
- Note that this results in the same payoffs as the second-price auction.

NE of first-price auction

- There are many NE, but in all of them, the winner is Player 1 (the player with the highest valuation of the object).
- Suppose the action profile is $(b_1, ..., b_n)$ and Player 1 does *not* win.
- $b_i > b_1$ for some $i \neq 1$.
- If $b_i > v_2$, then Player i's payoff is negative, so he has an incentive to deviate by bidding 0.
- If $b_i \leq v_2$, then Player 1 can increase his payoff from 0 to $v_1 = b_i$ by bidding b_i.
- Therefore, no such action profile is a NE.

Review of Random Variables

- A *random variable* is a variable that can take on different values, according to some probability distribution.
- A finite, *discrete* random variable is random variable that can take on a finite number of values.
- For example, suppose that y is a random variable that can take on two values:
  - $y = 1$ occurs with probability p,
  - $y = 0$ occurs with probability $1 - p$.
- y can represent the outcome of flipping a biased coin that shows 1 on one side and 0 on the other side.
- If we flip this coin a very large number of times, the *frequency* (i.e. the fraction of flips) of showing 1 will be p.

Expected Values

- The *expected value* of a random variable is the weighted sum of all possible outcomes, weighted by the probability of occurrence.
- For the biased coin example, the expected value is:

$$E(y) = Pr(y = 1) \cdot 1 + Pr(y = 0) \cdot 0$$

$$= p \cdot 1 + (1 - p) \cdot 0$$

$$= p$$

- In general, the expected value (EV) of a discrete random variable that can take on n outcomes $y_1, ...y_n$ with probabilities $p_1, ...p_n$ is

$$E(y) = p_1 y_1 + p_2 y_2 + ... + p_n y_n$$

- The sum of probabilities over all outcomes must add up to 1.

Next Lecture & Homework

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- For next week, please read Chapter 4.