CUR 412: Game Theory and its Applications, Lecture 6

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April 5, 2016

Announcements

- HW #2 is due next week.
- I will return HW #1 next week.
- The midterm will be in class on April 19. It will cover Chapters 1-4 (only the sections that we've gone over in lectures).
- Midterm will be closed-book. No programmable calculators or smartphones allowed.
- Previous midterms and solutions are on the course website.

Review of Last Week

- We introduced the concept of a *mixed strategy*: a *probability distribution* over a player's set of actions.
- Instead of players choosing an action, they choose a probability distribution over their set of actions.
- Players rank outcomes based on their *expected payoff*, using the probability distributions generated by all players' mixed strategies.
- A *mixed strategy Nash equilibrium* (MSNE) is a mixed strategy profile where no player can get a higher expected payoff by unilaterally changing his mixed strategy.

Review of Last Week

- A *pure strategy* is a mixed strategy where one action has probability 1, and all other actions have probability 0.
- This is equivalent to our previous concept of strategy in games without randomization.
- A game with mixed strategies can have more NE than the corresponding game without randomization (i.e. if we only allow pure strategies).
- We'll see that NE of the non-random game are a subset of the NE when mixed strategies are allowed.

Review of Last Week

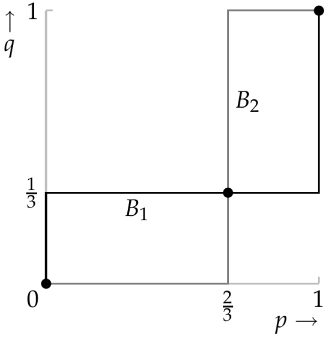
- Player i's *best response* function is similar as before; now it gives the set of mixed strategies (i.e. probability distributions) that give the highest expected payoff, conditional on the mixed strategies of all other players α_{-i}.
- In 2-player, 2-action games, we can calculate Player i's expected payoff to each of his actions (or equivalently, pure strategies). Suppose Player i's actions are $\{a, b\}$.
  - If Player i's expected payoff to action a, denoted $E_i(a)$, is strictly greater than $E_i(b)$, then the best response is the pure strategy $(\alpha(a) = 1, \alpha(b) = 0)$.
  - If $E_i(a) = E_i(b)$, then any mixed strategy $(\alpha(a) = p, \alpha(b) = 1 - p)$ for $0 \leq p \leq 1$ is a best response.

Example: BoS

	B	S
B	2,1	0,0
S	0,0	1,2

- Suppose p, q are the probabilities assigned to B by Player 1 and Player 2, respectively.
  - $E_1(B, \alpha_2) = 2 \cdot q + 0 \cdot (1 - q) = 2q$
  - $E_2(S, \alpha_2) = 0 \cdot q + 1 \cdot (1 - q) = 1 - q$
- If $2q > 1 - q \rightarrow q > 1/3$, unique best response is pure strategy B
- If $q < 1/3$, unique best response is pure strategy S
- If $q = 1/3$, all mixed strategies are best responses

BoS: Best Response Function



$$B_1(q) = \begin{cases} \{0\} & \text{if } q < 1/3 \\ \{p : 0 \le p \le 1\} & \text{if } q = 1/3 \\ \{1\} & \text{if } q > 1/3 \end{cases}$$

- There are two Nash equilibria in pure strategies, and one new equilibrium in mixed strategies.

The "Colonel Blotto" game

- There are two players, the Attacker and the Defender.
- Each player has two armies.
- There are two locations being defended. Each player allocates his 2 armies to the 2 locations.
- At each locations, the Defender wins if he has at least as many armies as the Attacker.
- The Defender wins the game if he wins at both locations.

The "Colonel Blotto" game

	(0,2)	(1,1)	(2,0)
(0,2)	-1,1	1,-1	1,-1
(1,1)	1,-1	-1,1	1,-1
(2,0)	1,-1	1,-1	-1,1

- Player 1 (the row player) is the Attacker.
- In each row and column, there is an outcome where Attacker wins, and an outcome where Defender wins.
- Therefore, there is no pure strategy NE, since the loser can always switch actions and become the winner.
- This can be generalized to any number of locations and number of armies for each player.

A Useful Characterization of Mixed Strategy NE

- So far, we've found mixed strategy NE by constructing best response functions.
- For more complicated games, this is too difficult. We'll state a condition that must hold true at any mixed strategy NE.
- A player's expected payoff to the mixed strategy α is a weighted average of his expected payoffs to playing each action:

$$U_i(\alpha) = \Sigma_{a_i} \alpha_i(a_i) E_i(a_i, \alpha_{-i})$$

- $E_i(a_i, \alpha_{-i})$ is the expected payoff of playing action a_i, when the other players use the mixed strategies α_{-i}.

A Useful Characterization of Mixed Strategy NE

$$U_i(\alpha) = \Sigma_{a_i} \alpha_i(a_i) E_i(a_i, \alpha_{-i})$$

- The value of a weighted average (with positive weights) must lie *between* the highest and lowest values.
- Suppose α^* is a mixed strategy NE, and Player i gets expected payoff E_i^* in this equilibrium.
- This must be in between the highest and lowest expected payoffs to actions that have a positive probability in α_i^*.
- Player i's expected payoff to all strategies (mixed and pure) is at most, E_i^*, by definition of MSNE.
- The highest expected payoff to an action cannot be higher than E_i^*, and the lowest cannot be lower.
- Therefore, all actions that have a positive probability in α_i^* must have the same expected payoff, equal to E_i^*.

Characterization of Mixed Strategy NE of a Finite Game

- **Proposition**: A mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if, for each player i:
  - The expected payoff (given other players' strategies α_{-i}) to every action in α_i^* with a positive probability, is the *same*, equal to $U_i(\alpha^*)$
  - The expected payoff (given other players' strategies α_{-i}) to every action in α_i^* with zero probability, is *at most*, equal to the expected payoff in the first condition
- We can use this condition to check whether some mixed strategy profile α is a mixed strategy NE.
- Check that the expected payoffs to each action in α_i with positive probability, is the same.

Example: BoS

	B	S
B	2,1	0,0
S	0,0	1,2

- In *BoS*, let's check the mixed strategy profile $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$.
- Given Player 2's mixed strategy $\alpha_2(B) = \frac{1}{3}, \alpha_2(S) = \frac{2}{3}$:
  - $E_1(B, \alpha_2) = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot 0 = \frac{2}{3}$
  - $E_1(S, \alpha_2) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$
- Given Player 1's mixed strategy $\alpha_1(B) = \frac{2}{3}, \alpha_1(S) = \frac{1}{3}$:
  - $E_2(B, \alpha_1) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3}$
  - $E_2(S, \alpha_1) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 2 = \frac{2}{3}$
- Therefore, this is a mixed strategy NE.

Example 117.1

	L	C	R
T	$\cdot, 2$	$3, 3$	$1, 1$
M	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
B	$\cdot, 4$	$5, 1$	$0, 7$

- Is the strategy pair $((\frac{3}{4}, 0, \frac{1}{4}), (0, \frac{1}{3}, \frac{2}{3}))$ a MSNE?
- The dots indicate irrelevant payoffs (they occur with zero probability).
- Given Player 2's mixed strategy: $\alpha_2(L) = 0, \alpha_2(C) = \frac{1}{3}, \alpha_2(R) = \frac{2}{3}$:
  - $E_1(T, \alpha_2) = 0 + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
  - $E_1(M, \alpha_2) = 0 + \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
  - $E_1(B, \alpha_2) = 0 + \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$
- T, B occur with positive probability in α_1, and have the same expected payoff when Player 2 plays α_2.
- M occurs with zero probability in α_1, and has an expected payoff not greater than the expected payoffs to T, B.

Example 117.1

	L	C	R
T	$\cdot, 2$	$3, 3$	$1, 1$
M	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
B	$\cdot, 4$	$5, 1$	$0, 7$

- Given Player 1's mixed strategy: $\alpha_1(T) = \frac{3}{4}, \alpha_1(M) = 0, \alpha_1(B) = \frac{1}{4}$:
  - $E_2(L, \alpha_2) = \frac{3}{4} \cdot 2 + 0 + \frac{1}{4} \cdot 4 = \frac{5}{2}$
  - $E_2(C, \alpha_2) = \frac{3}{4} \cdot 3 + 0 + \frac{1}{4} \cdot 1 = \frac{5}{2}$
  - $E_2(R, \alpha_2) = \frac{3}{4} \cdot 1 + 0 + \frac{1}{4} \cdot 7 = \frac{5}{2}$
- C, R occur with positive probability in α_2, and have the same expected payoff when Player 1 plays α_1.
- L occurs with zero probability in α_2, and has an expected payoff not greater than the expected payoffs to C, R.
- Note: the fact that $E_2(L, \alpha_2) = \frac{5}{2}$ does not imply anything about the existence of a MSNE that has a positive probability on L.

The "Colonel Blotto" game

	(0,2)	(1,1)	(2,0)
(0,2)	-1,1	1,-1	1,-1
(1,1)	1,-1	-1,1	1,-1
(2,0)	1,-1	1,-1	-1,1

- A MSNE is where both players choose mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- This case of the Colonel Blotto game has no other equilibrium, but other cases may have many equilibria.
- In general, it is a difficult problem to find *all* equilibria of a game.

Exercise 117.2: Choosing Numbers

- Players 1 and 2 choose a positive integer from $1...K$.
- If the players choose the same number, Player 2 gets a payoff of -1 and Player 1 gets a payoff of 1.
- Otherwise, both players get a payoff of 0.
- First, show that one MSNE is if both players choose each integer with equal probability $1/K$.

Exercise 117.2: Choosing Numbers

- Player 1's expected payoffs to his actions $1, ..., K$ are:
- $E_1(1) = E_1(2) = ... = E_1(K) = 1/K$
- Player 2's expected payoffs to his actions $1, ..., K$ are:
- $E_2(1) = E_2(2) = ... = E_2(K) = -1/K$
- All actions with positive probability have the same payoff, so the condition for a MSNE is satisfied.

Exercise 117.2: Choosing Numbers

- Show there is no other MSNE.
- Let Player 1's mixed strategy be $(p_1, ..., p_K)$.
- Let Player 2's mixed strategy be $(q_1, ..., q_K)$.
- Player 1's expected payoffs to his actions $1, ..., K$ are:
- $E_1(1) = q_1, E_1(2) = q_2, ..., E_1(K) = q_K$
- Player 2's expected payoffs to his actions $1, ..., K$ are:
- $E_2(1) = -p_1, E_2(2) = -p_2, ..., E_2(K) = -p_K$

Exercise 117.2: Choosing Numbers

- Player 1's expected payoff, given both player's mixed strategies, is:

$$p_1 q_1 + p_2 q_2 + \ldots + p_K q_K$$

- Player 2's expected payoff, given both player's mixed strategies, is the negative of Player 1's expected payoff:

$$-p_1 q_1 - p_2 q_2 - \ldots - p_K q_K$$

- Suppose that Player 1 does not place equal probability on each number: there exists a number i such that p_i that is strictly greater than the other p's.
- Player 2's expected payoff to playing i is $-p_i$, so Player 2 will put zero probability on $i : q_i = 0$.
- However, if $q_i = 0$, then Player 1's expected payoff to i is 0, and Player 1's best response is to put zero probability on i, a contradiction.

Implications of Mixed Strategy NE

- A MSNE α^* that is not a pure strategy equilibrium is never *strict*. Player *i* is indifferent between α_i^* and the actions that have a positive probability in α_i^*.
- In MSNE, Player *i*'s probabilities are such that they induce the *other* players to become indifferent among their actions.

Existence of Equilibrium in Finite Games

- Does every finite game have a MSNE?
- A famous result, proved by Nash, shows that this is true.
- **Proposition**: Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium.
- We won't prove this, but if you are interested, it uses Kakutani's fixed point theorem.

Dominated Actions

- Recall: an action a_1 strictly dominates a_2 if it gives a higher payoff, no matter what other players do.
- We can extend this to mixed strategies.
- **Definition**: In a strategic game with vNM preferences, player i's mixed strategy α_i **strictly dominates** action a'_i if:

$$U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i}) \quad \text{for every list of the other players' actions } a_{-i}$$

- Note that the other players' actions are pure strategies.
- It is possible for an action that is not strictly dominated by a pure strategy, to be strictly dominated by a mixed strategy.

Example of Dominated Actions

	L	R
T	1	1
M	4	0
B	0	3

- T is not strictly dominated by M or B.
- But, it is strictly dominated by $\alpha_1(T) = 0, \alpha_1(M) = \frac{1}{2}, \alpha_1(B) = \frac{1}{2}$.
  - If Player 2 plays L: expected payoff is $\frac{1}{2} \cdot 4 = 2$
  - If Player 2 plays R: expected payoff is $\frac{1}{2} \cdot 3 = 1.5$

Strictly Dominated Actions in Mixed Strategy NE

- In a pure strategy Nash equilibrium, no player uses a strictly dominated action.
- Extend to mixed strategies: In a mixed strategy Nash equilibrium, no player will place a positive probability on a strictly dominated action.

Weak Domination

- **Definition**: A mixed strategy α_i **weakly dominates** action a_i' if

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i}) \quad \text{for every list } a_{-i} \text{ of the other players' actions}$$

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \quad \text{for some } a_{-i} \text{ of the other players' actions}$$

- We say action a_i' is *weakly dominated*.
- As before, a weakly dominated action may be played with positive probability in a mixed strategy NE.

Pure Equilibria when Randomization is Allowed

- The pure version of Bos had 2 NE. These were also equilibria in the game where mixing was allowed, which had 3 NE. This is true in general.
- Suppose G is a strategic game without randomization. Preferences are represented by the payoff function u_i.
- Suppose G' is a strategic game with randomization, that has the same players and actions as G, and preferences are represented by expected values of u_i.
- If a^* is a NE of G, then the mixed strategy profile in which each player assigns probability 1 to action a_i^* is also a mixed strategy NE of G'.
- If α^* is a mixed strategy NE of G' in which all players assign probability 1 to an action a_i, then the action profile $(a_1 ... a_n)$ is a Nash equilibrium of G.

Illustration: Reporting a Crime

- Suppose a crime is observed by a group of n people.
- Each person would like the police to be informed, but prefers that someone else makes the phone call.
- Specifically, each person gains value v from the police being informed, but pays cost c if she calls.
- Game with vNM preferences:
  - Players: n people
  - Actions: Each player chooses to $\{Call, Don'tCall\}$
  - Preferences: Each player i has expected value preferences over a payoff function that gives 0 if no one calls, $v - c$ if player i calls, and v if someone other than player i calls.

Nash Equilibrium in Pure Strategies

- Nash equilibrium in pure strategies: exactly one player will call.
- Case 1: only Player i calls. Let's check this is a Nash equilibrium.
  - Player i gets payoff $v - c$, everyone else gets payoff v.
  - Player i can switch to *Don't call*, but payoff will be lowered from $v - c$ to 0.
  - All other players can switch to *Call*, but payoff will be lowered from v to $v - c$.
- Case 2: All players choose *Don't call*.
  - Any player can switch to *Call*, payoff increases from 0 to $v - c$.
- Case 3: Two or more players call. Suppose players i, j call.
  - Either player can switch to *Don't call*, increase payoff from $v - c$ to v.

Symmetric Mixed Strategy Nash Equilibrium

- There is no symmetric (i.e. all players choose the same action) NE in pure strategies.
- However, there is one in mixed strategies. Suppose that all players place probability p on *Call*, $1 - p$ on *Don't call*.
- Equilibrium condition: EV to all actions with positive probability are equal, given other players' mixed strategies α_{-i}.
- EV to *Call*, given α_{-i} : $v - c$
- EV to *Don't call*, given α_{-i} :

$$\text{Prob(nobody else calls)} \cdot 0 + \text{Prob(at least 1 person calls)} \cdot v = v - c$$

$$(1 - \text{Prob(nobody else calls)}) \cdot v = v - c$$

$$\rightarrow \frac{c}{v} = \underbrace{\text{Prob(nobody else calls)}}_{(1-p)^{n-1}}$$

Symmetric Mixed Strategy Nash Equilibrium

$$\frac{c}{v} = (1-p)^{n-1} \rightarrow p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

- What happens as n increases?
- p decreases, so the equilibrium probability that any given player will call goes down to zero as $n \rightarrow \infty$.
- The probability that no one calls is:

$$(1-p)^n = (1-p)^{n-1} \cdot (1-p) = \frac{c}{v}(1-p)$$

- So as n increases, the equilibrium probability that at least one person calls also goes down to zero as $n \rightarrow \infty$.

Next week

- Please finish Chapter 4 and read Chapter 5.1-5.2.
- HW #2 is due next week.
- I will return HW #1 next week.
- The midterm will be in class on April 19. It will cover Chapters 1-4 (only the sections that we've gone over in lectures).
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