CUR 412: Game Theory and its Applications, Lecture 7

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April 12, 2015

Announcements

- The midterm will be **next week** in class. It will be closed-book and cover Chapters 1-4 (only the sections that we've gone over in lectures).
- Previous midterms and solutions are on the course website.
- No smartphones or programmable calculators will be allowed. Ordinary calculators are fine.
- I'm returning HW #1.
  - HW # 1: questions 1-3 are 1 point, the rest are 2 points
- HW #2 is due at the end of class today.

Review of Last Week

- **Proposition**: A mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if, for each player i:
  - The expected payoff (given other players' strategies α_{-i}) to every action in α_i^* with a positive probability, is the *same*, equal to $U_i(\alpha^*)$
  - The expected payoff (given other players' strategies α_{-i}) to every action in α_i^* with zero probability, is *at most*, equal to the expected payoff in the first condition
- We can use this condition to check whether some mixed strategy profile α is a mixed strategy NE.
- Check that the expected payoffs to each action in α_i with positive probability, is the same.

Review of Last Week

- Suppose G is a finite, strategic form game without randomization, and let G' be the same game, where randomization is allowed and players have expected payoff preferences.
- G' has at least one MSNE.
- The NE of G (if they exist) are also pure strategy MSNE's of G'.
- The pure strategy MSNE's of G' are NE of G.
- G' may have other non-pure mixed strategy NE.

Symmetric Mixed Strategy Equilibrium

- Before, we saw that a pure strategy Nash equilibrium could be interpreted as a *steady state* of a social situation, in which players are randomly drawn from populations (one population for each type of player).
- A *symmetric* game is a social situation where each player is facing the *same* situation.
- Examples:
  - Two pedestrians walking towards each other
  - Two cars approaching each other at an intersection
  - (from last week) People who have observed a crime
  - Commuters deciding whether to drive a car, or take the subway (congestion games)
- In all of these situations, the role of each player is identical to every other player.

Symmetric Mixed Strategy Equilibrium

- The problem facing a player in a symmetric game, is caused by other people exactly like yourself
- We can interpret an equilibrium of this game as a steady state of a social situation, where all players are randomly drawn from the *same* population.
- A pure strategy symmetric equilibrium is a steady state where all players choose the same action. (e.g. drive on the right)
- A mixed strategy symmetric equilibrium is a steady state where fractions of the players choose different actions (corresponding to the probability of each action)

Symmetric Mixed Strategy Equilibrium

	L	R
L	w,w	x,y
R	y,x	z,z

- In a 2-player, 2-action symmetric game, the payoffs must have this structure.
- Both players have the same actions, and the payoffs must be identical if Player 1 and Player 2's roles are exchanged.
- A symmetric pure-strategy Nash equilibrium is a NE where both players choose the same action.
- We can extend this concept to a symmetric MSNE: a mixed-strategy NE where all players choose the same mixed strategy.

Approaching Pedestrians

	L	R
L	1,1	0,0
R	0,0	1,1

- This symmetric game has two symmetric pure-strategy NE.
- There is also a symmetric MSNE, where each player chooses L with probability $\frac{1}{2}$.
- Note that the MSNE makes players worse off than the pure strategy NE, since there is a collision half of the time.
- When players prefer to choose the *same* action, we say that their interests coincide.

	X	Y
X	0,0	1,1
Y	1,1	0,0

- This symmetric game has no symmetric pure-strategy NE.
- It also has a symmetric MSNE, where each player chooses X with probability $\frac{1}{2}$.
- In this game, players' interests do not coincide, since they prefer to choose different actions.

Existence of Symmetric MSNE

- Similar to the case for MSNE, it can be proven that a symmetric MSNE must exist in a finite symmetric game.
- **Proposition 130.1**: Every symmetric strategic game with vNM preferences in which each player's set of actions is finite, has a symmetric mixed strategy NE.

Exercise 118.3: Defending Territory

- There are two generals, *Attacker* and *Defender*.
- *Attacker* has 2 armies, *Defender* has 3 armies.
- There are two locations that must be defended.
- Each general must choose how many armies to allocate to each location.
- In each location, if *Defender* has at least as many armies as *Attacker*, then *Defender* wins the battle at that location.
- *Defender* wins the game if he wins at both locations.

Exercise 118.3: Defending Territory

Attacker

Defender	(0, 2)	(1, 1)	(2, 0)
(0, 3)	1,0	0,1	0,1
(1, 2)	1,0	1,0	0,1
(2, 1)	0,1	1,0	1,0
(3, 0)	0,1	0,1	1,0

- No pure Nash equilibria.
- Note that for *Defender*, $(1, 2)$ weakly dominates $(0, 3)$ and $(2, 1)$ weakly dominates $(3, 0)$.

Exercise 118.3: Defending Territory

- First, we show that *Attacker* will not play $(1,1)$ in any MSNE.
- Let q_1, q_2, q_3 denote *Attacker*'s probability of playing $(0,2)$, $(1,1)$, $(2,0)$ respectively, with $q_1 + q_2 + q_3 = 1$.
- *Attacker* must play $(0,2)$ and $(2,0)$ with positive probability; otherwise, *Defender* has an action that guarantees victory.
- Suppose *Attacker* plays $(1,1)$ with positive probability ($q_2 > 0$).
- *Defender*'s expected payoff to his pure strategies are:
  - $(0,3) : q1$
  - $(1,2) : q1 + q2$
  - $(2,1) : q2 + q3$
  - $(3,0) : q3$

Exercise 118.3: Defending Territory

- *Defender*'s expected payoff to his pure strategies are:
  - $(0,3) : q1$
  - $(1,2) : q1 + q2$
  - $(2,1) : q2 + q3$
  - $(3,0) : q3$
- *Defender*'s best response is to place zero probability on $(0,3)$ and $(3,0)$, since $(1,2)$ and $(2,1)$ give a strictly higher expected payoff.
- But if *Defender* does not play $(0,3)$ and $(3,0)$, then $(1,1)$ becomes strictly dominated for *Attacker*, and therefore cannot be played in equilibrium.
- This is a contradiction, so the assumption must be false: *Attacker* does not play $(1,1)$ with positive probability.

Exercise 118.3: Defending Territory

- If we assume *Attacker* will not play $(1, 1)$, then *Defender*'s actions:
  - $(0, 3)$ and $(1, 2)$ have the same payoffs
  - $(2, 1)$ and $(3, 0)$ have the same payoffs
- *Attacker* will choose q_1, q_3 to make *Defender* indifferent between his actions that are played with positive probability.
- *Defender*'s expected payoff to playing $(0, 3)$ or $(1, 2)$: q_1
- *Defender*'s expected payoff to playing $(2, 1)$ or $(0, 3)$: q_3
- Equalizing them gives $q_1 = q_3 = 0.5$.

Exercise 118.3: Defending Territory

- Now, consider *Defender*'s strategy. Assume the probability placed on $(0,3)$, $(1,2)$, $(2,1)$, $(0,3)$ are p_1, p_2, p_3, p_4 respectively, where $p_1 + p_2 + p_3 + p_4 = 1$.
- We can eliminate *Attacker*'s action $(1,1)$.
- For *Defender*, $(0,3)$ is equivalent in payoff to $(1,2)$, so for a given level of $p_1 + p_2$, the values of p_1 and p_2 are irrelevant.
- Likewise, $(2,1)$ is equivalent in payoff to $(0,3)$, so for a given level of $p_3 + p_4$, the values of p_3 and p_4 are irrelevant.
- The conditions for MSNE must be satisfied: $E_{Attacker}(1,1) \leq E_{Attacker}(0,2) = E_{Attacker}(2,0)$

Exercise 118.3: Defending Territory

- The conditions for MSNE must be satisfied: $E_{Attacker}(1,1) \leq E_{Attacker}(0,2) = E_{Attacker}(2,0)$
  - $E_{Attacker}(0,2) = p_3 + p_4$
  - $E_{Attacker}(1,1) = p_1 + p_4$
  - $E_{Attacker}(2,0) = p_1 + p_2$

$$E_{Attacker}(0,2) = E_{Attacker}(2,0) \Rightarrow p_1 + p_2 = p_3 + p_4$$

- Since $p_1 + p_2 + p_3 + p_4 = 1$, then $p_1 + p_2 = p_3 + p_4 = 0.5$.

$$E_{Attacker}(1,1) \leq E_{Attacker}(0,2) \Rightarrow p_1 + p_4 \leq p_3 + p_4 \Rightarrow p_1 \leq p_3$$

$$E_{Attacker}(1,1) \leq E_{Attacker}(2,0) \Rightarrow p_1 + p_4 \leq p_1 + p_2 \Rightarrow p_4 \leq p_2$$

Exercise 118.3: Defending Territory

- A mixed strategy profile is a MSNE if:
- *Attacker* plays $(0, 2)$, $(2, 0)$ with probability 0.5 on each
- *Defender* plays p_1, p_2, p_3, p_4 such that:

$$p_1 + p_2 = p_3 + p_4 = 0.5$$

$$p_1 \leq p_3, p_4 \leq p_2$$

What Does It Mean To Play A Mixed Strategy?

- The concept of mixed strategy Nash equilibrium makes some assumptions that may or may not hold in the real world.
- First, players are assumed to know the mixed strategies of other players. How do they know this?
- If we assume a situation that is repeatedly played many times, then we can observe the frequency of each action.
- However, in a one-shot situation, there are no past examples to learn from.
- We can interpret the mixed strategy of other players as a *belief* about their behavior, rather than an empirical frequency.
- We'll examine this idea later in Chapter 9.

What Does It Mean To Play A Mixed Strategy?

- Second, do people really randomize their actions in important situations?
- Again, this is plausible in repeated situations, but perhaps not in one-shot situations.
- It can be argued that in games with conflict, I want my actions to be unpredictable by the other player.
- However, in games with cooperation (e.g. BoS), it is beneficial to me if the other player can predict my actions.
- Mixed strategies can be interpreted as applying to populations, instead of individuals.

Extensive Form Games (Chapter 5)

- So far, we've been using strategic form (or normal form) games. All players are assumed to move simultaneously.
- This cannot capture a sequential situation, where one player moves, then another...
- Or, if one player can get information on the moves of the other players, before making his own move.
- We will introduce a way of specifying a game that allows this.

Example: An Entry Game

- Suppose we have a situation where there is an *incumbent* and a *challenger*.
- For example, an industry might have an established dominant firm.
- A challenger firm is deciding whether it wants to enter this industry and compete with the incumbent.
- If the challenger enters, the incumbent chooses whether to engage in intense (and possibly costly) competition, or to accept the challenger's entry.

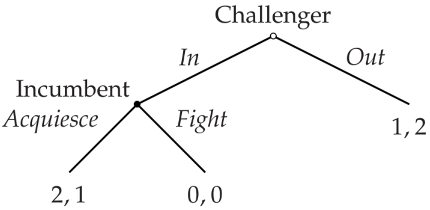
Entry Game

- There are two players: the incumbent and the challenger.
- The challenger moves first, has two actions: *In* and *Out*.
- If the challenger chooses *In*, the incumbent chooses *Fight* or *Acquiesce*.
- Challenger's preference over outcomes: $(In, Acquiesce) > (Out) > (In, Fight)$
- Incumbent's preference over outcomes: $(Out) > (In, Acquiesce) > (In, Fight)$
- We can represent these preferences with the payoff functions (challenger is u_1):

$$u_1(In, Acquiesce) = 2, u_1(Out) = 1, u_1(In, Fight) = 0$$

$$u_2(Out) = 2, u_2(In, Acquiesce) = 1, u_2(In, Fight) = 0$$

Game Tree



- We can represent this game with a tree diagram.
- The root node of the tree is the first move in the game (here, by the challenger).
- Each action at a node corresponds to a branch in the tree.
- Outcomes are leaf nodes (i.e. there are no more branches).
- The first number at each outcome is the payoff to the first player (the challenger).

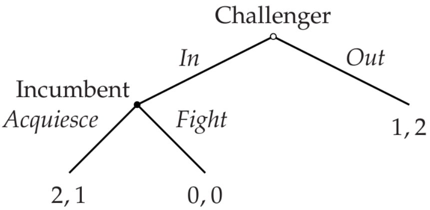
Formal Specification of an Extensive Game

- Formally, we need to specify all possible sequences of actions, and all possible outcomes.
- A *history* is the sequence of actions played from the beginning, up to some point in the game.
  - In the tree, a history is a path from the root to some node in the tree.
  - In the entry game, all possible histories are: $\varnothing$ (i.e. at the beginning, no actions played yet), $(In), (Out), (In, Acquiesce), (In, Fight)$.
- A *terminal history* is a sequence of actions that specifies an outcome, which is what players have preferences over.
  - In the tree, a terminal history is a path from the root to a leaf node (a node with no branches).
  - In the entry game, the terminal histories are: $(Out), (In, Acquiesce), (In, Fight)$.
- A *player function* specifies whose turn it is to move, at every non-terminal history (every non-leaf node in the tree).

Formal Specification of an Extensive Game

- An extensive game is specified by four components:
  - A set of **players**
  - A set of **terminal histories**, with the property that no terminal history can be a subsequence of some other terminal history
  - A **player function** that assigns a player to every non-terminal history
  - For each player, **preferences** over the set of terminal histories
- The sequence of moves and the set of actions at each node are implicitly determined by these components.
- In practice, we will use trees to specify extensive games.

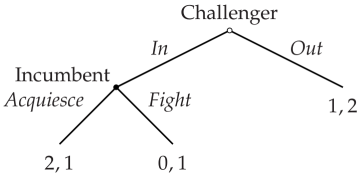
Solutions to Entry Game



- How can we find the solution to this game?
- First approach: Each player will imagine what will happen in future nodes, and use that to determine his choice in current nodes.
- Suppose we're at the node just after the challenger plays *In*.
- At this point, the payoff-maximizing choice for the incumbent is *Acquiesce*, which gives a payoff pair (2,1).
- So, at the beginning, the challenger might assume playing *In* gives a payoff pair of (2,1), which gives a higher payoff than *Out*.
- This approach is called *backwards induction*: imagining what will happen at the end, and using that to determine what to do in earlier situations.

Backwards Induction

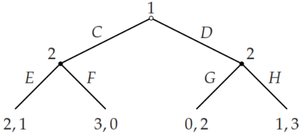
- At each move, for each action, a player deduces the actions that all players will rationally take in the future.
- This gives the outcome that will occur (assuming everyone behaves rationally), and therefore gives the payoff to each current action.
- However, in some cases, backwards induction doesn't give a clear prediction about what will happen.



- In this version of the Entry Game, both *Acquiesce*, *Fight* give the same payoff to the incumbent. Unclear what to believe at the beginning of the game.
- Also, games with infinitely long histories (e.g. an infinitely repeating game).

Strategies in Extensive Form Games

- Another approach is to formulate this as a strategic game, then use the Nash equilibrium solution concept.
- We need to expand the action sets of the players to take into account the different actions at each node.
- For each player i, we will specify the action chosen at all of i's nodes, i.e. every history after which it's i's turn to move
- **Definition**: A **strategy** of player i in an extensive game with perfect information is a function that assigns to each history h after which it is i's turn to move, an action in $A(h)$ (the actions available after h).



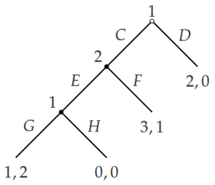
- In this game, Player 1 only moves at the start (i.e. after the empty history $\varnothing$). The actions available are C, D, so Player 1 has two strategies: $\varnothing \to C, \varnothing \to D$.
- Player 2 moves after the history C and also after D. After C, available actions are E, F. After D, available actions are G, H.
- Player 2 has four strategies:

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H

- In this case, it's simple enough to write them together. We can refer to these strategies as EG, EH, FG, FH. The first action corresponds to the first history C.

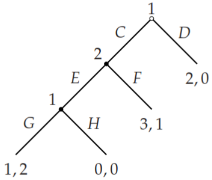
Strategies in Extensive Form Games

- We can think of a strategy as an action plan or contingency plan: If Player 1 chooses action X, do Y.
- However, a strategy must specify an action for *all* histories, even if they do not occur due to previous choices in the strategy.



- In this example, a strategy for Player 1 must specify an action for the history (C, E), even if it specifies D at the beginning.
- Think of this as allowing for the possibility of mistakes in execution.

Strategy Profiles & Outcomes



- As before, a *strategy profile* is a list of the strategies of all players.
- Given a strategy profile s, the terminal history that results by executing the actions specified by s is denoted $O(s)$, the *outcome* of s.
- For example, in this game, the outcome of the strategy pair (DG, E) is the terminal history D.
- The outcome of (CH, E) is the terminal history (C, E, H).

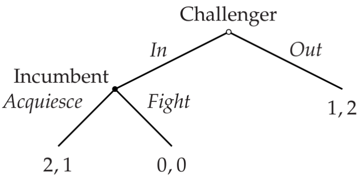
Nash Equilibrium

- **Definition** The strategy profile s^* in an extensive game with perfect information is a **Nash equilibrium** if, for every player i and strategy r_i of player i, the outcome $O(s^*)$ is at least as good as the outcome $O(r_i, s^*_{-i})$ generated by any other strategy profile (r_i, s^*_{-i}) in which player i chooses r_i:

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i$$

- We can construct the *strategic form* of an extensive game by listing all strategies of all players and finding the outcome.

Strategic Form of Entry Game



- The strategic form of the Entry Game is:

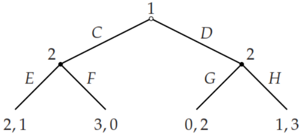
		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

- There are two Nash equilibria: $(In, Acquiesce)$ and $(Out, Fight)$.
- The first NE is the same as the one found with backwards induction.
- In the second NE, the incumbent chooses *Fight*. However, if *In* is taken as given, this is not rational. This is called an *incredible threat*.
- If the incumbent could commit to *Fight* at the beginning of the game, it would be credible.

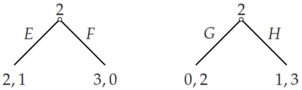
Subgames

- The concept of Nash equilibrium ignores the sequential structure of an extensive game.
- It treats strategies as choices made once and for all at the beginning of the game.
- However, the equilibria of this method may contain incredible threats.
- We'll define a notion of equilibrium that excludes incredible situations.
- Suppose Γ is an extensive form game with perfect information.
- The *subgame* following a non-terminal history h, $\Gamma(h)$, is the game beginning at the point just after h.
- A *proper subgame* is a subgame that is not Γ itself.

Subgames

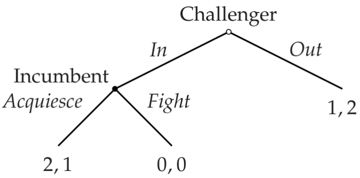


- This game has two proper subgames:



Subgame Perfect Equilibria

- A *subgame perfect equilibrium* is a strategy profile s^* in which each subgame's strategy profile is also a Nash equilibrium.
- Each player's strategy must be optimal for all subgames that have him moving at the beginning, not just the entire game.



- $(Out, Fight)$ is a NE, but is not a subgame perfect equilibrium because in the subgame following *In*, the strategy *Fight* is not optimal for the incumbent.

Subgame Perfect Equilibria

- Every subgame perfect equilibrium is also a Nash equilibrium, but not vice versa.
- A subgame perfect equilibrium induces a Nash equilibrium in every subgame.
- In games with finite histories, subgame perfect equilibria are consistent with backwards induction.

Next Week: Midterm

- The midterm will be **next week** in class. It will be closed-book and cover Chapters 1-4 (only the sections that we've gone over in lectures).
- Previous midterms and solutions are on the course website.
- No cellphones or programmable calculators will be allowed. Ordinary calculators are fine.