CUR 412: Game Theory and its Applications, Lecture 8

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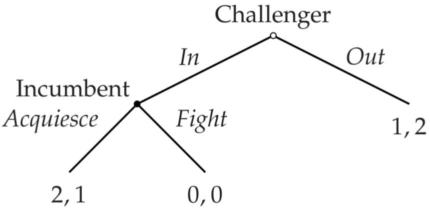
April 26, 2016

Announcements

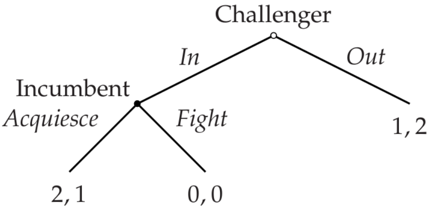
- HW #3 will be posted later today, due in 2 weeks.

Review of Previous Lecture: Extensive Games



- Extensive games can model a situation where players move sequentially.
- We describe extensive games with a *game tree*.
- Each *node* in the tree corresponds to a player's turn to move.
- Each branch from a node corresponds to an action of the player who moved.

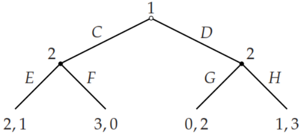
Review of Previous Lecture: Extensive Games



- A *history* is a sequence of actions (i.e. a path from the *root node* to some other node).
- A *terminal history* is a path from the root node to a leaf node (i.e. an "ending" to the game).
- Players have preferences over terminal histories.

Strategies in Extensive Form Games

- One approach to solving an extensive form game is to formulate it as a strategic (i.e. simultaneous-move) game, then use the Nash equilibrium solution concept.
- We need to expand the definition of an "strategy".
- Before, a player's strategy only had to specify one action.
- Now, it must specify the player's action at *all* places in the game where that player moves.
- **Definition**: A **strategy** of player i in an extensive game with perfect information is a function that assigns to each history h after which it is i's turn to move, an action in $A(h)$ (the actions available after h).



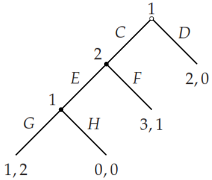
- In this game, Player 1 only moves at the start (i.e. after the empty history $\varnothing$). The actions available are C, D, so Player 1 has two strategies: $\varnothing \to C, \varnothing \to D$.
- Player 2 moves after the history C and also after D. After C, available actions are E, F. After D, available actions are G, H.
- Player 2 has four strategies:

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H

- In this case, it's simple enough to write them together. We can refer to these strategies as EG, EH, FG, FH. The first action corresponds to the first history C.

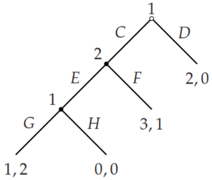
Strategies in Extensive Form Games

- We can think of a strategy as an action plan or contingency plan: If Player 1 chooses action X, do Y.
- However, a strategy must specify an action for *all* histories, even if they do not occur due to previous choices in the strategy.



- In this example, a strategy for Player 1 must specify an action for the history (C, E), even if it specifies D at the beginning.
- Think of this as allowing for the possibility of mistakes in execution.

Strategy Profiles & Outcomes



- As before, a *strategy profile* is a list of the strategies of all players.
- Given a strategy profile s, the terminal history that results by executing the actions specified by s is denoted $O(s)$, the *outcome* of s.
- For example, in this game, the outcome of the strategy pair (DG, E) is the terminal history D.
- The outcome of (CH, E) is the terminal history (C, E, H).

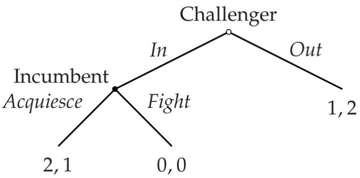
Nash Equilibrium

- **Definition** The strategy profile s^* in an extensive game with perfect information is a **Nash equilibrium** if, for every player i and strategy r_i of player i, the outcome $O(s^*)$ is at least as good as the outcome $O(r_i, s^*_{-i})$ generated by any other strategy profile (r_i, s^*_{-i}) in which player i chooses r_i:

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i$$

- We can construct the *strategic form* of an extensive game by listing all strategies of all players and finding the outcome.

Strategic Form of Entry Game



- The strategic form of the Entry Game is:

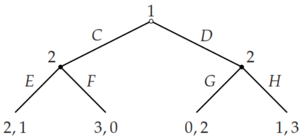
		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

- There are two Nash equilibria: $(In, Acquiesce)$ and $(Out, Fight)$.
- The first NE is the same as the one found with backwards induction.
- In the second NE, the incumbent chooses *Fight*. However, if *In* is taken as given, this is not rational. This is called an *incredible threat*.
- If the incumbent could commit to *Fight* at the beginning of the game, it would be credible.

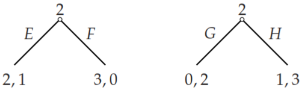
Subgames

- The concept of Nash equilibrium ignores the sequential structure of an extensive game.
- It treats strategies as choices made once and for all at the beginning of the game.
- However, the equilibria of this method may contain incredible threats.
- We'll define a notion of equilibrium that excludes incredible situations.
- Suppose Γ is an extensive form game with perfect information.
- The *subgame* following a non-terminal history h, $\Gamma(h)$, is the game beginning at the point just after h.
- A *proper subgame* is a subgame that is not Γ itself.

Subgames

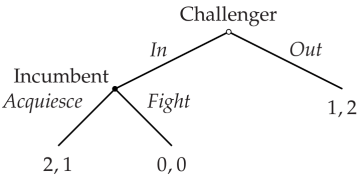


- This game has two proper subgames:



Subgame Perfect Equilibria

- A *subgame perfect equilibrium* is a strategy profile s^* in which each subgame's strategy profile is also a Nash equilibrium.
- Each player's strategy must be optimal for all subgames that have him moving at the beginning, not just the entire game.



- $(Out, Fight)$ is a NE, but is not a subgame perfect equilibrium because in the subgame following *In*, the strategy *Fight* is not optimal for the incumbent.

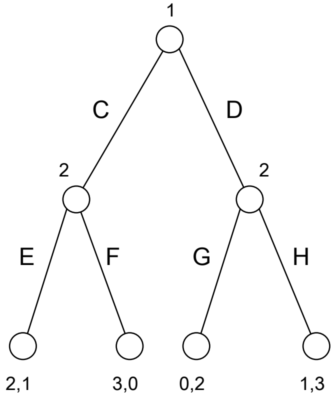
Subgame Perfect Equilibria

- Every subgame perfect equilibrium is also a Nash equilibrium, but not vice versa.
- A subgame perfect equilibrium induces a Nash equilibrium in every subgame.
- In games with finite histories, subgame perfect equilibria are consistent with backwards induction.

Backwards Induction in Finite-Horizon Games

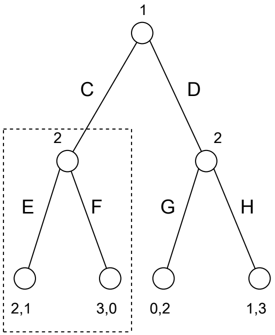
- In a game with a *finite horizon* (i.e. finite maximum length of all terminal histories), we can find all SPNE through backwards induction.
- This procedure can be interpreted as reasoning about how players will behave in future situations.
- Procedure:
  - For all subgames of length 1 (i.e. 1 action away from a terminal node), find the optimal actions of the players.
  - Take these actions as given. For all subgames of length 2, find the optimal actions of the players...
  - Repeat until we cover the entire tree.

Example 170.1



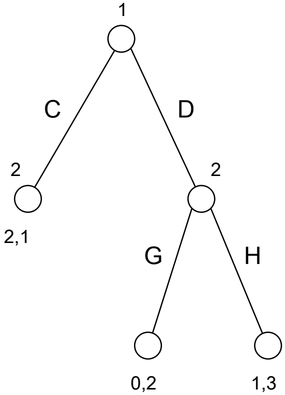
- There are 2 subgames with length 1.

Example 170.1



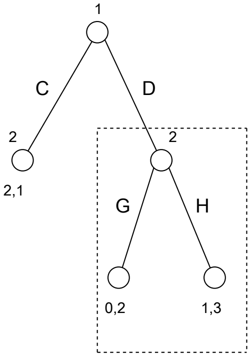
- Consider the left subgame. It is Player 2's turn to move.
- Player 2's optimal action is E, resulting in payoff $(2, 1)$.
- We will assume Player 2 always chooses E, so the payoff of this subgame is $(2, 1)$.

Example 170.1



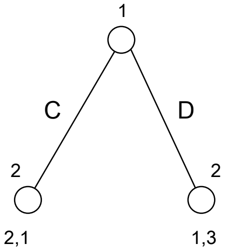
- Therefore, the payoff to Player 1 choosing C is $(2, 1)$.

Example 170.1



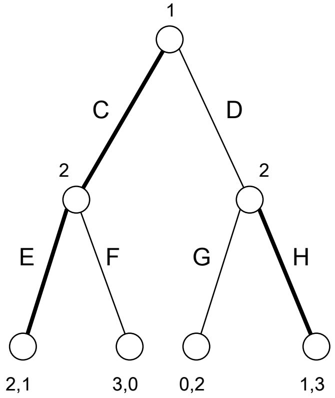
- Consider the right subgame. It is Player 2's turn to move.
- Player 2's optimal action is H, resulting in payoff $(1, 3)$.
- We will assume Player 2 always chooses H, so the payoff of this subgame is $(1, 3)$.

Example 170.1



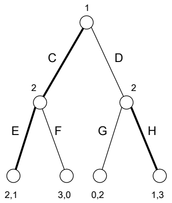
- Therefore, the payoff to Player 1 choosing D is $(1, 3)$.
- Now, Player 1's optimal action is C.
- Backwards induction gives the strategy pair (C, EH).
- The outcome of (C, EH) is the terminal history CE with payoff $(2, 1)$.

Example 170.1



- We mark the optimal actions at each node with thick lines.

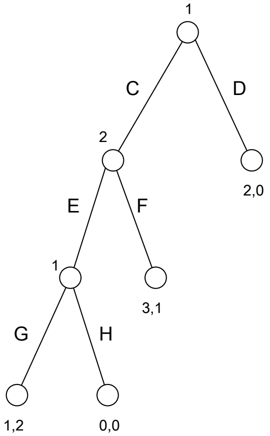
Strategic Form of 170.1



	EG	EH	FG	FH
C	2,1	2,1	3,0	3,0
D	0,2	1,3	0,2	1,3

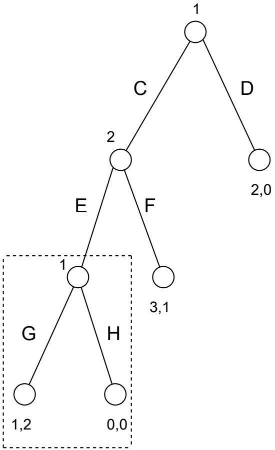
- Let's compare the backwards induction result $(C, EH)$to the NE of the strategic form.
- (C, EG) and (C, EH) are NE of strategic form.
- However, (C, EG) includes a non-optimal action for Player 2 in the right subgame, so is not a subgame-perfect NE.

Example 160.1



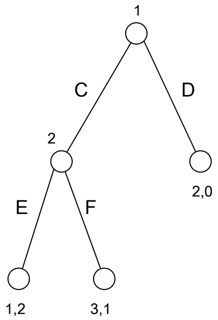
- There is one subgame with length 1, and one subgame with length 2.

Example 160.1



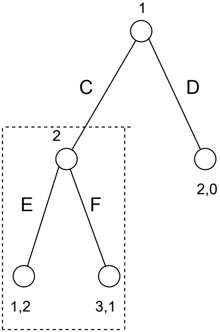
- In this subgame, it is Player 1's turn to move.
- Optimal action is G, resulting in payoff $(1, 2)$.

Example 160.1



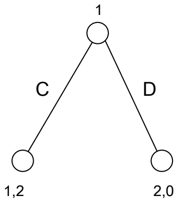
- Assume Player 1 chooses G with certainty.
- Then, the payoff to choosing E is $(1, 2)$.

Example 160.1



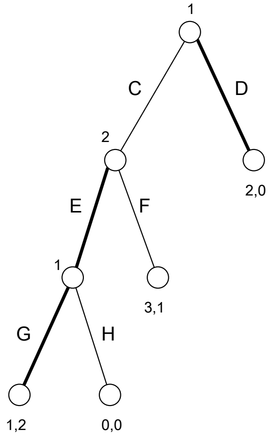
- In this subgame, it is Player 2's turn to move.
- Optimal action is E, resulting in payoff $(1, 2)$.

Example 160.1



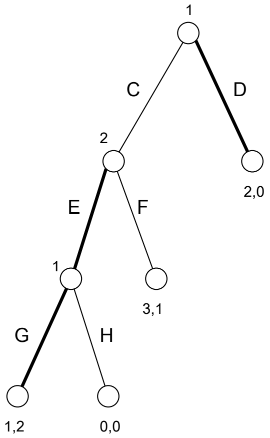
- Optimal action is D, resulting in payoff $(2, 0)$.
- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.

Example 160.1



- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.

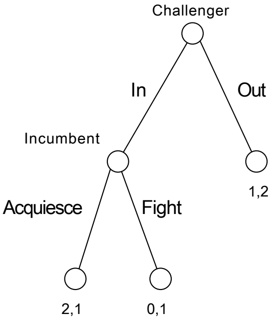
Strategic Form of 160.1



	E	F
CG	1,2	3,1
CH	0,0	3,1
DG	2,0	2,0
DH	2,0	2,0

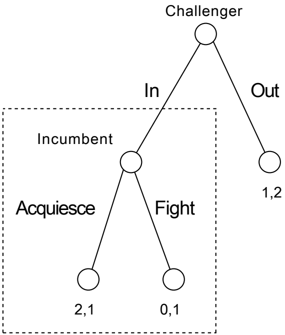
- NE of strategic form are: $(CH, F), (DG, E), (DH, E)$.
- Only (DG, E) is a subgame perfect NE.

Example 158.1 (Variant of Entry Game)



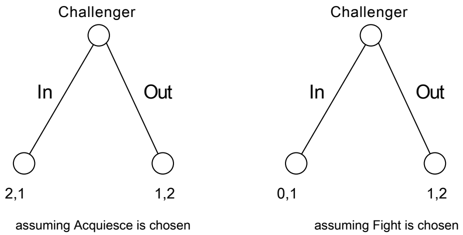
- What if there are multiple optimal actions in a subgame? Then we need to keep track of them separately.
- This is a variant of the entry game in which the Incumbent is indifferent between *Acquiesce*, *Fight*.

Example 158.1 (Variant of Entry Game)



- In this subgame, both *Acquiesce* and *Fight* are optimal actions.
- We cannot eliminate either as an irrational choice. So, we keep track of both possibilities.

Example 158.1 (Variant of Entry Game)

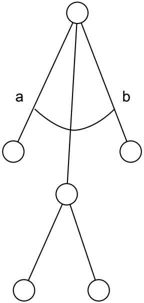


- Backwards induction gives $(In, Acquiesce)$ and $(Out, Fight)$.
- In this case, the NE of the strategic form are the same as the subgame-perfect NE.

Continuous Action Sets

- The action set at a node may be infinite (e.g. if the player chooses a real number).
- In this case, we graphically represent this with an arc between the lowest and highest possible values.
- Effectively, there are an *infinite* number of branches in the game tree at this node.
- Suppose it is Player i's turn to move after all of these branches. Then Player i's strategy profile must specify an action for *all* possible branches.

Continuous Action Sets

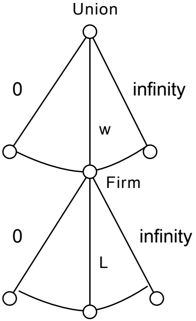


- If the infinite set of actions is an interval of real numbers $[a, b]$, then Player i's strategy profile for this node must be a *function* over $[a, b]$.
- For a strategy profile to be a subgame perfect NE, it must induce a NE at each of the infinite subgames.

Exercise 177.1 (Firm-Union Bargaining)

- A union and a firm are bargaining.
- First, the union presents a wage demand $w \geq 0$.
- The firm chooses an amount $L \geq 0$ of labor to hire.
- The firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labor, and 2500 if $L > 50$.
- The price of output is 1.
- The firm's preferences are represented by its profits.
- The union's preferences are represented by the total wage bill, wL.

Exercise 177.1 (Firm-Union Bargaining)



- The firm's payoff is its profit, given by:

$$\Pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- Union's payoff: wL

Exercise 177.1 (Firm-Union Bargaining)

$$\Pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- For every $w \geq 0$, there is a subgame where the firm's payoff depends on w.
- Profit has a quadratic part (if $L \leq 50$) and a linear part (if $L > 50$), and is continuous at $L = 50$.
- We want to find the profit-maximizing choice of L.
- The linear part is decreasing in L, so we can ignore it (its maximum is at $L = 50$).
- Quadratic part is maximized at $L^* = \frac{100-w}{2}$.

Exercise 177.1 (Firm-Union Bargaining)

- Quadratic part is maximized at $L^* = \frac{100-w}{2}$.
- Firm's profit is:

$$\frac{100-w}{2}(100 - \frac{100-w}{2}) - w\frac{100-w}{2} = \frac{(w-100)^2}{4}$$

- Profit is always non-negative. Firm's best response correspondence is:

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

Exercise 177.1 (Firm-Union Bargaining)

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

- Now, consider the union's decision.
- If $w = 0$ or $w > 100$, union's payoff is 0.
- $wB_f(w) = \frac{w(100-w)}{2}$ is maximized at $w^* = 50$.
- $L^* = B_f(50) = 25$.

Exercise 177.1 (Firm-Union Bargaining)

- The set of subgame perfect NE is:
- Union's strategy profile: at the empty history, choose $w = 50$.
- Firm's strategy profile: at the subgame following the history w, choose an element of $B_f(w)$.
- Note that the firm has an infinite number of strategy profiles, but there is only one equilibrium outcome, since only the subgame after $w = 50$ will be realized.
- Firm's payoff is 625 and union's payoff is 1250.

Exercise 177.1 (Firm-Union Bargaining)

- Is there an outcome that both players prefer to the SPNE outcome with payoffs $(1250, 625)$?
- Suppose that instead of each player maximizing his own payoff, a *social planner* could choose both w and L.
- The sum of payoffs is $L(100 - L) - wL + wL = L(100 - L)$ which is maximized at $L = 50$. The choice of w then allocates payoffs to the firm and union.
- For example, if $w = 30$, then the firm's payoff is 1000 and the union's payoff is 1500.
- This is an illustration that individual maximization may not achieve the most efficient outcome.

Exercise 177.1 (Firm-Union Bargaining)

- Is there a Nash equilibrium outcome that differs from any subgame perfect NE outcome?
- Suppose the union's strategy is: offer $w = 100$ and the firm's strategy profile is: for any w, offer $L = 0$.
- The firm has no incentive to deviate, since it will make a negative payoff for any $L > 0$.
- The union has no incentive to deviate, because it will get a payoff of 0 for any choice of w.
- This is not subgame perfect, since the firm's strategy is not optimal for $w < 100$.

Characteristics of Finite Horizon Games

- **Proposition**: In a finite horizon extensive game with perfect information, the set of strategy profiles isolated by backwards induction is the set of all subgame-perfect equilibria.
- **Proposition**: Every finite extensive game with perfect information has a subgame perfect equilibrium.

Announcements

- Please read Chapter 5.
- Also, HW #3 will be posted later today, due in 2 weeks.