CUR 412: Game Theory and its Applications, Lecture 9

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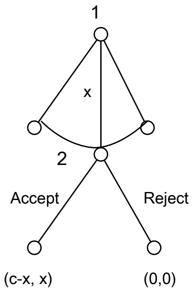
Announcements

- HW #3 is due next week.

Ch. 6.1: Ultimatum Game

- This is a simple game that can model a very simplified bargaining situation.
- Two people want to split some amount $c > 0$. The procedure is as follows:
  - First, Player 1 offers $0 \geq x \geq c$ to Player 2.
  - Then, Player 2 chooses to *Accept* or *Reject* the offer.
  - If he chooses *Accept*, payoffs are: $c - x$ for Player 1 and x for Player 2.
  - If he chooses *Reject*, both players get 0.

Ultimatum Game



- Suppose $c = 5$. I will choose three people to be Player 1, and three people to be Player 2.
- Player 1 will write down an offer between 0 and 5.
- Player 2 will write down *Accept* or *Reject*.

SPNE of Ultimatum Game

- We can use backwards induction to find the SPNE of this game.
- Consider the last subgame. Taking x as given, Player 2 will optimally *Accept* if $x > 0$, and may choose either *Accept* or *Reject* if $x = 0$.
- Two possible strategies for Player 2:
  - Player 2 *Accept*s all offers $x \geq 0$, and *Rejects* all other offers
  - Player 2 *Accept*s all offers $x > 0$, and *Rejects* all other offers

SPNE of Ultimatum Game

- Player 1's decision: consider each of Player 2's possible strategies separately.
- If Player 2 *Accepts* all offers $x \geq 0$:
  - Player 1's optimal offer is $x = 0$.
  - Player 2 will *Accept*, leading to payoffs $(c, 0)$.
- If Player 2 only *Accepts* offers $x > 0$:
  - There is no optimal offer for Player 1: it is better to offer x as close as possible to 0 while still being > 0.
  - Therefore, the second strategy cannot be part of a SPNE.
- We are left with a single SPNE: Player 1 offers $x = 0$, and Player 2 accepts all offers $x \geq 0$.

Experimental Testing of the Ultimatum Game

- The SPNE solution concept makes a very clear prediction: Player 1 offers zero, and Player 2 accepts all offers $x \geq 0$.
- However, when people play the Ultimatum Game in experiments, they consistently choose a different result.
- In experiments, Player 1 offers around $0.3c$. Player 2 chooses *Reject* about 20% of the time.
- Why the difference? Two possible explanations:
  - Equity: real people also value equity or "fairness", but the players in the model only care about their own payoff.
  - Repeated interactions: in real life, people interact repeatedly, so Player 2 can choose *Reject* to develop a reputation for punishing low offers. In the game, there is only one interaction.

Exercise 185.2: Fair Division of a Cake

- Suppose two people want to divide a cake into two pieces such that both people will be satisfied, *without* asking a third person to divide it.
- Suppose the total cake has size=1.
  - Player 1 cuts the cake into two pieces (chooses a number x between 0 and 1);
  - Player 2 chooses the piece he prefers (chooses either x or $1-x$).
  - Each player's payoff is equal to the size of the piece they receive.

Exercise 185.2: Fair Division of a Cake

- Player 2's decision: suppose x has been chosen by Player 1.
- If $x < \frac{1}{2}$, best response is $1 - x$.
- If $x > \frac{1}{2}$, best response is x.
- If $x = \frac{1}{2}$, either is a best response (let's suppose that Player 2 chooses x).
- Assuming this strategy of Player 2, then Player 1's payoff as a function of x is:
  - If $x < \frac{1}{2}$, payoff will be x.
  - If $x \geq \frac{1}{2}$, payoff will be $1 - x$.
- Player 1's best response is $x = \frac{1}{2}$.

Stackelberg Duopoly

- In Chapter 3, we saw models of duopoly (Cournot and Bertrand) that were simultaneous-move situations.
- Now, let's look at a duopoly model where one firm gets to move first. Does being first to move provide an advantage or disadvantage?
- Suppose firms are competing by choosing output quantities, as in Cournot duopoly.
- Each of two firms $i = 1, 2$ chooses to produce $q_i \geq 0$ units.
- Total cost of production: $C_i(q_i)$.
- Inverse market demand function: $P_d(Q)$, where $Q = q_1 + q_2$.
- Profits: $P_d(q_i + q_j)q_i - C_i(q_i)$.

Stackelberg Duopoly

- Assume:
  - constant unit cost: $C_i(q_i) = cq_i$
  - market demand:

$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$$

- Best response of firm 2, taking q_1 as given, is:

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \leq \alpha - c \\ 0 & \text{if } q_1 > \alpha - c \end{cases}$$

- This will be Firm 2's strategy in the last subgame.

Stackelberg Duopoly

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \leq \alpha - c \\ 0 & \text{if } q_1 > \alpha - c \end{cases}$$

- Firm 1's problem is to choose q_1 that maximizes

$$q_1 P(q_1 + b_2(q_1)) - cq_1 = q_1(\alpha - c - (q_1 + \frac{1}{2}(\alpha - c - q_1))) = \frac{1}{2} q_1(\alpha - c - q_1)$$

- This is a quadratic that is maximized at $q_1 = \frac{1}{2}(\alpha - c)$.
- Unique SPNE outcome: Firm 1 chooses $q_1 = \frac{1}{2}(\alpha - c)$
- Firm 2's strategy is given by $b_2(q_1)$.
- The outcome for Firm 2 will be $q_2 = b_2(q_1) = b_2(\frac{1}{2}(\alpha - c)) = \frac{1}{4}(\alpha - c)$.

Stackelberg Duopoly

- Compared to the Cournot duopoly outcome where both firms choose $q_i = \frac{1}{3}(\alpha - c)$, Firm 1 produces more and Firm 2 produces less.
- Firm 1's profits are also higher: $\frac{1}{8}(\alpha - c)^2$ compared to $\frac{1}{9}(\alpha - c)^2$ in Cournot duopoly.
- In this case, going first is an advantage for the first mover.

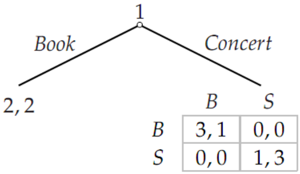
Ch. 7.1: Allowing for Simultaneous Moves

- In the extensive games we've studied so far, there are no simultaneous moves: after every history (i.e. at every node in the game tree), only one player chooses an action.
- We can generalize this to allow multiple players to move simultaneously at a node, just as they do in a strategic game.
- The definition of an extensive game does not need to be changed, except that the player function (that assigns a player to every history, i.e. the player whose turn it is to move) can now return a *set* of players.

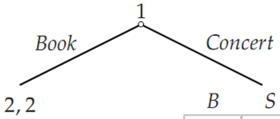
Variant of BoS

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

- Suppose that before this game of *BoS* takes place, Player 1 makes a choice between *Book* and *Concert*.
- If *Book* is chosen, the payoff is $(2, 2)$. If *Concert* is chosen, then both players are faced with a *BoS* game.
- This game can be described by this tree:



Variant of BoS



	B	S
B	3, 1	0, 0
S	0, 0	1, 3

- Just as before, we can convert this game to strategic form by listing all strategies (i.e. all actions at every point where it is the player's turn to move) and writing down the payoffs.

	B	S
(Concert, B)	3, 1	0, 0
(Concert, S)	0, 0	1, 3
(Book, B)	2, 2	2, 2
(Book, S)	2, 2	2, 2

Variant of BoS

	B	S
$(Concert, B)$	$3, 1$	$0, 0$
$(Concert, S)$	$0, 0$	$1, 3$
$(Book, B)$	$2, 2$	$2, 2$
$(Book, S)$	$2, 2$	$2, 2$

- The Nash equilibria are $((Concert, B), B), ((Book, B), S)$, and $((Book, S), S)$.
- To find the subgame perfect equilibria, we can check each NE if it satisfies the subgame perfect condition: the strategy must also be a NE in every subgame.
- Here, the subgame is BoS, and we know the NE for BoS are (B, B) and (S, S).
- Therefore, only $((Concert, B), B)$ and $((Book, S), S)$ are SPNE.

Variant of BoS

- We could also find the SPNE with backwards induction.
- Starting from the *BoS* subgame, the equilibrium action profiles are (B, B) and (S, S).
- As in the case where there was more than one optimal action in the subgame, we need to keep track of all NE action profiles.
  - Suppose the NE of the subgame is (B, B). Then the payoff to Player 1 of choosing *Concert* is 3, so the optimal action is *Concert* and the SPNE is $((Concert, B), B)$.
  - Suppose the NE of the subgame is (S, S). Then the payoff to Player 1 of choosing *Concert* is 1, so the optimal action is *Book* and the SPNE is $((Book, S), S)$.

Ch 7.2: Entry into a monopolized industry

- Consider a model that combines the Entry Game with Cournot Duopoly.
- There is an industry which is currently monopolized by the *Incumbent* firm.
- A *Challenger* firm is deciding whether to choose *In* or *Out*.
- If *In* is chosen, *Challenger* has to pay a cost of entry f; then both firms face Cournot duopoly.
- If *Out* is chosen, *Challenger* makes zero profit while *Incumbent* faces a monopoly problem.

Ch 7.2: Entry into a monopolized industry

- Assume the same parameters as in the Cournot duopoly we've seen before:
- Cost of production $C_i(q_i) = cq_i$, i.e. constant marginal cost, equal for both firms.
- Inverse market demand is given by:

$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } \alpha \geq Q \\ 0 & \text{if } \alpha < Q \end{cases}$$

Subgame after *In*

- Consider the subgame if *In* is chosen. We know the outcome of the Cournot duopoly game in this case is:
- $q_{incumbent} = q_{challenger} = (\alpha - c)/3$. Both firms choose the same output
- *Incumbent*'s profit is: $\pi_{incumbent} = \frac{1}{9}(\alpha - c)^2$
- *Challenger*'s profit is: $\pi_{challenger} = \frac{1}{9}(\alpha - c)^2 - f$

Subgame after *Out*

- Consider the subgame if *Out* is chosen. This is a standard monopoly problem where the *Incumbent* firm chooses q to maximize

$$\pi(q) = q(\alpha - q) - cq$$

- The optimal value of q is $(\alpha - c)/2$.
- *Incumbent*'s profit is $\frac{1}{4}(\alpha - c)^2$
- *Challenger*'s profit is 0.

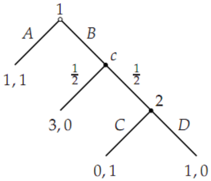
Challenger's first move

- Subgame after *In*:
  - *Incumbent*'s profit: $\pi_{incumbent} = \frac{1}{9}(\alpha - c)^2$
  - *Challenger*'s profit: $\pi_{challenger} = \frac{1}{9}(\alpha - c)^2 - f$
- Subgame after *Out*:
  - *Incumbent*'s profit is: $\pi_{incumbent} = \frac{1}{4}(\alpha - c)^2$
  - *Challenger*'s profit is: $\pi_{challenger} = \frac{1}{9}(\alpha - c)^2 - f$
- *Challenger* will choose *In* if $\frac{1}{9}(\alpha - c)^2 - f > 0$ and *Out* if $\frac{1}{9}(\alpha - c)^2 - f < 0$.
- If they are equal, there are two SPNE outcomes, one where *Challenger* chooses *In* and one where it chooses *Out*.

Ch 7.6: Allowing for exogenous uncertainty

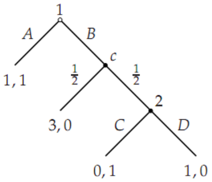
- So far, all the extensive games we've seen have been deterministic: no randomness in outcomes.
- We can allow randomness by introducing an additional player, called "Chance" or "Nature"
- Nature makes choices randomly, according to a known probability distribution.
- Players' preferences are now over lotteries.
- As before, we will assume von Neumann-Morgenstern preferences (i.e. lotteries are valued by their expected payoff).

Ch 7.6: Allowing for exogenous uncertainty



- Consider this game with chance moves.
- Here, "c" is the Chance player, who chooses randomly between two branches, each with probability $1/2$.
- This is still a game with *perfect information*: at each player's move, he knows exactly what sequence of moves has occurred in the past.
- In the last subgame, Player 2 chooses C, so this subgame has a payoff of $(0, 1)$.

Ch 7.6: Allowing for exogenous uncertainty

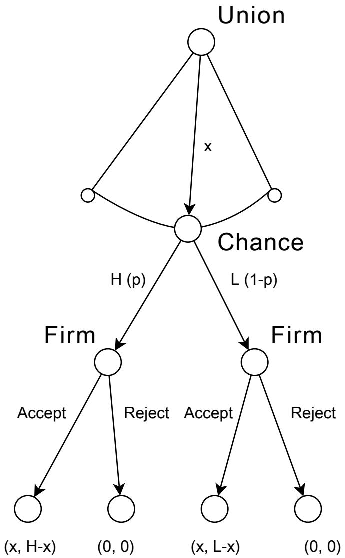


- In the subgame following B, Chance chooses each action with probability $\frac{1}{2}$.
- The expected payoff to Player 1 of this subgame is therefore $\frac{1}{2}3 + \frac{1}{2}0 = 1.5$.
- The expected payoff to Player 2 of this subgame is $\frac{1}{2}0 + \frac{1}{2}1 = 0.5$.
- At the beginning, Player 1 chooses B, since the expected payoff of B is greater than the expected payoff of A.

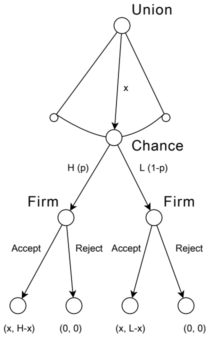
Exercise 227.2: Firm-Union Bargaining

- Suppose a firm and a union are bargaining over how to split the "surplus", i.e. the profits before labor has been paid.
- Surplus is a random variable, that takes on the value H with probability p, and $L < H$ with probability $1 - p$.
- The sequence of the game is as follows:
  - First, the union (who does not know the outcome of the surplus, just its distribution) makes a demand $x \geq 0$.
  - The firm, who does know what the surplus (denoted z) is, can choose to *Accept* or *Reject*. If *Accept* is chosen, the union gets x and the firm gets $z - x$. If *Reject* is chosen, both players get 0.

Exercise 227.2: Firm-Union Bargaining

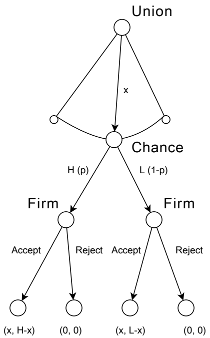


Exercise 227.2: Firm-Union Bargaining



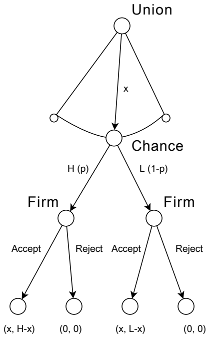
- Note that "Chance" moves after the union but before the firm; this is how we ensure that the firm makes its choice with the knowledge of the outcome of the surplus.
- Again, note that this game has perfect information.
- Assume that in case of a tie in payoffs, the firm chooses *Accept*.

Exercise 227.2: Firm-Union Bargaining



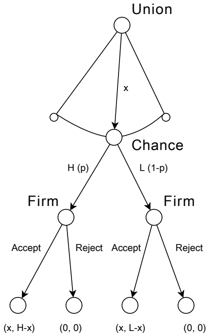
- We can solve this with backwards induction:
  - In the bottom left subgame, take x as given. The firm will choose *Accept* if $H - x \geq 0$.
  - In the bottom right subgame, take x as given. The firm will choose *Accept* if $L - x \geq 0$.

Exercise 227.2: Firm-Union Bargaining



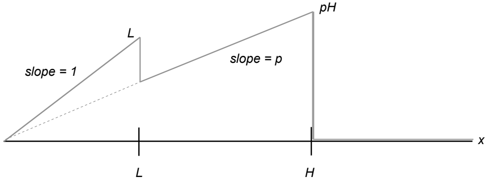
- Taking the firm's actions as given, the union chooses x. Let's consider the possible cases for x:
- Suppose $x \leq L$. Then $x < H$ as well, so the firm will *Accept* if either H or L is the outcome. The expected payoff to the union is: $px + (1 - p)x = x$.

Exercise 227.2: Firm-Union Bargaining



- Suppose $L < x \leq H$. The firm will *Accept* if H is the outcome, but will *Reject* if L is the outcome. The expected payoff is: $px + (1 - p)0 = px$.
- Suppose $H < x$. The firm will always *Reject*, so the expected payoff is 0.

Exercise 227.2: Firm-Union Bargaining



- We can plot the expected payoff as a function of x.
- If $L > pH$, then $x = L$ is the optimal action for the union.
- If $L < pH$, then $x = H$ is the optimal action.
- If $L = pH$, then either is optimal for the union, and there is more than one SPNE outcome.

Next Week

- HW # 3 is due next week.
- For next week, please read Chapter 7.