CUR 412: Game Theory and its Applications

Final Exam Solutions

Q1. **(16 pts.)** Answer whether each of the following statements is True or False. You don't have to provide an explanation.

(a) Every subgame perfect Nash equilibrium is a Nash equilibrium.

True. By definition, a subgame perfect NE is a strategy profile that induces a NE in every subgame, including the proper subgame (i.e. the original game).

(b) In the Stackelberg duopoly game, the follower becomes better off than in the Cournot game, since she can move after observing the leader's strategy.

False. In Stackelberg-Cournot duopoly, the leader is better off.

(c) If a static game is played repeatedly, then some outcome other than static Nash equilibria can possibly be achieved as a subgame perfect Nash equilibrium.

True. The subgame perfect folk theorem (Chapter 14.11) for the repeated Prisoner's Dilemma states that any feasible pair of payoffs in which each player gets at least $u_i(D, D)$ can be supported as a SPNE, if δ is close enough to 1.

(d) A finite extensive game with perfect information may fail to have a subgame perfect equilibrium.

False. An extensive game with a finite number of actions and a finite horizon will have at least one optimal action at each subgame, therefore a subgame perfect equilibrium must exist. If the action space is continuous, however, an optimal action may not exist.

Q2. **(20 pts.)** Consider the following game played between a boss (B) and an employee (E). The boss offers a wage, $w \geq 0$. After observing the wage offer, the employee decides how much effort, $e \geq 0$, to expend. The payoff functions for the boss and the employee are (α is a constant, $\alpha \geq 0$):

$$u_B(w, e) = 2\sqrt{e} - w$$

$$u_E(w, e) = w - \frac{e^2}{2} + \alpha we$$

(a) **(10 pts.)** What is the optimal effort for the employee, as a function of w?

The employee chooses $e \geq 0$ to maximize

$$u_E(w, e) = w - \frac{e^2}{2} + \alpha we$$

This is a downwards facing quadratic. The first-order condition is:

$$\frac{\partial u_E}{\partial e} = -e + \alpha w = 0$$

Therefore, the maximizing value of e as a function of w is $e^*(w) = \alpha w$.

(b) **(10 pts.)** What are the subgame perfect equilibrium choices of w and e, as a function of α?

Given $e^*(w) = \alpha w$, the boss chooses w to maximize

$$2\sqrt{\alpha w}$$

The first-order condition is:

$$\frac{\partial u_B}{\partial w} = \sqrt{\frac{\alpha}{w}} - 1 = 0$$

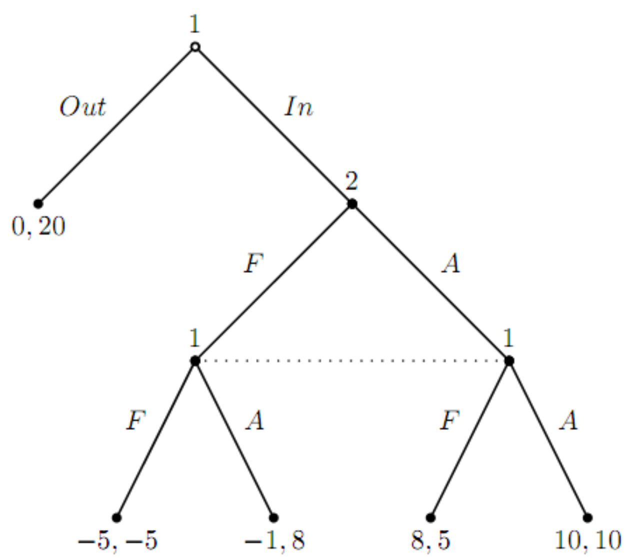
We verify this is a maximum by checking the second-order condition:

$$\frac{\partial^2 u_B}{\partial^2 w} = -\frac{\alpha^{\frac{1}{2}} w^{\frac{-3}{2}}}{2} < 0$$

The optimal choice of w is $w^* = \alpha$. Therefore, the unique SPNE is:

$$e^*(w) = \alpha^2, w^* = \alpha$$

Q3. **(20 pts.)** Find the set of pure strategy Nash equilibria and subgame perfect equilibria of the following game:



The strategic form of this game is as follows:

	F	A
$OutF$	0,20	0,20
$OutA$	0,20	0,20
InF	-5,-5	8,5
InA	-1,8	10,10

There are three pure-strategy NE: $(OutF, F)$, $(OutA, F)$, (InA, A).

To find the subgame perfect equilibria, first consider the subgame that follows In. It is a simultaneous-move game; we can write its strategic form as follows:

	F	A
F	-5,-5	8,5
A	-1,8	10,10

There is one pure-strategy NE: (A, A), resulting in payoffs $(10, 10)$. Player 1's optimal choice at the beginning is therefore In. There is a unique SPNE (InA, A).

Q4. **(24 pts.)** Consider the infinitely repeated version of the following game:

	C	D
C	4,4	0,6
D	6,0	1,1

The payoff of player i to any infinite sequence of payoffs $\{u_{it}\}$ is given by the normalized discounted sum of payoffs:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_{it}$$

where $0 < \delta < 1$.

(a) **(12 pts.)** For what values of δ, if any, does it constitute a subgame perfect equilibrium when both players choose this strategy?

- Choose C in period 1.
- Choose C after any history in which the previous period's outcome was (C, C).
- Choose D after any other history.

If a strategy profile is a SPNE, it must satisfy the "one-deviation property": there is no one-shot deviation (i.e. deviate for one period, then revert to the strategy profile) that gives a higher payoff. We will check all possible histories to find the conditions under which there is a profitable one-shot deviation.

Let's divide all possible histories into two types: histories that are either empty or ended in (C, C), and everything else.

- **Empty history, or (C, C) was previously played**: If both players do not deviate:
  - outcome path: $(C, C), (C, C), ...$
  - payoffs: $(4, 4), (4, 4), ...$
  - Both players' discounted average is 4.

  Suppose Player 1 does a one-shot deviation.

  - outcome path: $(D, C), (D, D), (D, D)...$

- payoffs: $(6,0),(1,1),(1,1),...$
- Player 1's discounted average: $(1-\delta)(6+\delta+\delta^2+...) = 6(1-\delta)+\delta$

Not deviating is optimal if $4 \geq 6(1-\delta)+\delta$, or if $\delta \geq \frac{2}{5}$.

- **All other histories**: If both players do not deviate:
  - outcome path: $(D,D),(D,D),...$
  - payoffs: $(1,1),(1,1),...$
  - Both players' discounted average is 1.

  Suppose Player 1 does a one-shot deviation.
  - outcome path: $(C,D),(D,D),(D,D)...$
  - payoffs: $(0,6),(1,1),(1,1),...$
  - Player 1's discounted average: δ, which is always smaller than 1.

Therefore, this is a SPNE if $\delta \geq \frac{2}{5}$.

(b) **(12 pts.)** For what values of δ, if any, does it constitute a subgame perfect equilibrium when both players choose this strategy?

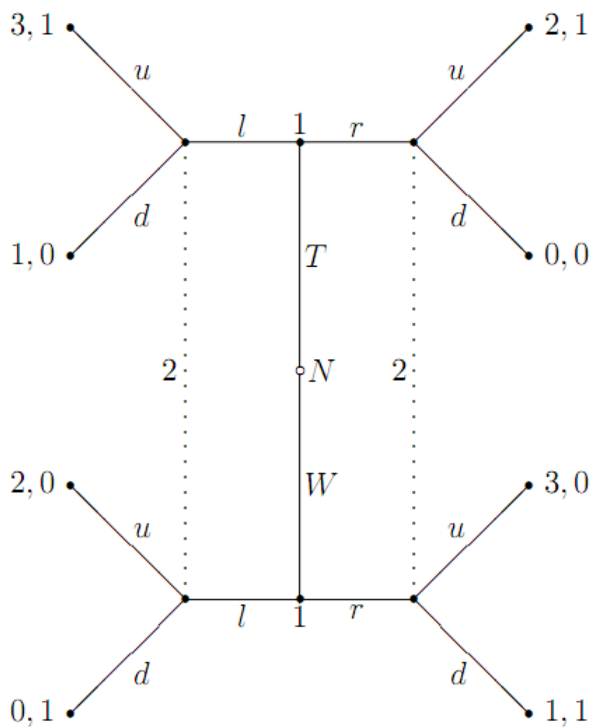
- Choose C in period 1.
- Do whatever your opponent did in the previous period.

This is Tit-for-Tat. We can divide up histories into four categories:

- ***ending in (C,C)***: Not deviating gives outcome path $(C,C),(C,C),...$, which results in a discounted average payoff of 4 to both players. Suppose Player 1 does a one-shot deviation, which gives outcome path $(D,C),(C,D),(D,C),(C,D)...$ Player 1 gets payoffs $6,0,6,0,...$ with a discounted average of $\frac{6}{1+\delta}$. Not deviating is optimal if $4 \geq \frac{6}{1+\delta}$, or if $\delta \geq \frac{1}{2}$.
- ***ending in (C,D)***: Not deviating gives outcome path $(D,C),(C,D),(D,C),(C,D),...$. Player 1 gets payoff sequence $6,0,6,0,...$ with a discounted average of $\frac{6}{1+\delta}$. If player 1 does a one-shot deviation, the outcome path is $(C,C),(C,C),...$ with a discounted average payoff of 4. Not deviating is optimal if $4 \leq \frac{6}{1+\delta}$, or if $\delta \leq \frac{1}{2}$.
- ***ending in (D,C)***: Not deviating gives outcome path $(C,D),(D,C),(C,D),(D,C),...$. Player 1 gets payoff sequence $0,6,0,6,...$ with a discounted average of $\frac{6\delta}{1+\delta}$. If Player 1 does a one-shot deviation, the outcome path is $(D,D),(D,D),...$ with a discounted average payoff of 1. Not deviating is optimal if $\frac{6\delta}{1+\delta} \geq 1$, or if $\delta \geq \frac{1}{5}$.
- ***ending in (D,D)***: Not deviating gives outcome path $(D,D),(D,D),...$ Player 1 gets a discounted average payoff of 1. If Player 1 does a one-shot deviation, the outcome path is $(C,D),(D,C),(C,D),(D,C),...$ Player 1 gets payoff sequence $0,6,0,6,...$ with a discounted average of $\frac{6\delta}{1+\delta}$. Not deviating is optimal if $1 \geq \frac{6\delta}{1+\delta}$, or if $\delta \leq \frac{1}{5}$.

There is no δ that satisfies all these conditions, so this strategy profile is not a SPNE.

Q5. **(20 pts.)** Consider the following signaling game. Nature (N) chooses the type of player 1 to be Tough (T) with probability 0.8, or Weak (W) with probability 0.2. Player 1 observes his type and chooses l or r. Player 2 observes only the action choice of player 1 but not the type, and chooses u or d. All these and the payoffs are common knowledge. Find the set of perfect Bayesian equilibria of this game.



First, let's find the set of pure strategy NE, then find the set of beliefs that would satisfy the requirements for a weak sequential equilibrium. The players' pure strategies and expected payoffs are shown in this matrix:

	$(U\|L,U\|R)$	$(U\|L,D\|R)$	$(D\|L,U\|R)$	$(D\|L,D\|R)$
LL	$0.8TLU+0.2WLU$	$0.8TLU+0.2WLU$	$0.8TLD+0.2WLD$	$0.8TLD+0.2WLD$
LR	$0.8TLU+0.2WRU$	$0.8TLU+0.2WRD$	$0.8TLD+0.2WRU$	$0.8TLD+0.2WRD$
RL	$0.8TRU+0.2WLU$	$0.8TRD+0.2WLU$	$0.8TRU+0.2WLD$	$0.8TRD+0.2WLD$
RR	$0.8TRU+0.2WRU$	$0.8TRD+0.2WRD$	$0.8TRU+0.2WRU$	$0.8TRD+0.2WRD$

	$(U\|L,U\|R)$	$(U\|L,D\|R)$	$(D\|L,U\|R)$	$(D\|L,D\|R)$
LL	2.8,0.8	2.8,0.8	0.8,0.2	0.8,0.2
LR	3,0.8	2.6,1	1.4,0	1,0.2
RL	2,0.8	0.4,0	1.6,1	0,0.2
RR	2.2,0.8	0.2,0.2	2.2,0.8	0.2,0.2

For Player 1, the first letter is the action played if T occurs. So, (LR) means: On T, choose

L; on W, choose R. For Player 2, the first letter is the action played if L occurs. So, (UD) means: On L, choose U; on R, choose D.

There are two pure NE: (LL, UD) and (RR, DU). Let's look at what beliefs must be for a weak sequential equilibrium. Let $p_{TL}, 1 - p_{TL}$ be Player 2's beliefs at the information set following L, and let $p_{TR}, 1 - p_{TR}$ be beliefs at the information set following R. The first number in each pair is the belief that Player 1 is type T.

- (LL, UD): By Bayes' Rule, p_{TL} must be 0.8, and p_{TR} is unrestricted, since the information set after R is not reached. At this information set, Player 2's expected payoff to playing U is $p_{TR}1 + (1 - p_{TR})0 = p_{TR}$; expected payoff to playing D is $p_{TR}0 + (1 - p_{TR})1 = 1 - p_{TR}$. For UD to be optimal, this requires that $1 - p_{TR} \geq p_{TR}$, or $p_{TR} \leq \frac{1}{2}$.
- (RR, DU): p_{TR} must be 0.8, and p_{TL} is unrestricted. At the information set after L, Player 2's expected payoff to playing U is $p_{TL}1 + (1 - p_{TL})0 = p_{TL}$; expected payoff to playing D is $p_{TL}0 + (1 - p_{TL})1 = 1 - p_{TL}$. For DU to be optimal, this requires that $1 - p_{TL} \geq p_{TL}$, or $p_{TL} \leq \frac{1}{2}$.

So the set of pure-strategy weak sequential equilibria are:

- $(LL, UD), p_{TL} = 0.8, 0 \leq p_{TR} \leq \frac{1}{2}$
- $(RR, DU), 0 \leq p_{TL} \leq \frac{1}{2}, p_{TR} = 0.8$