CUR 412: Game Theory and its Applications

Midterm Solutions

Q1. **(10 pts)**. Answer whether each of the following statements is True or False. You don't have to provide an explanation.

For the following questions, assume we have a 2-player game with a finite number of actions, and we have written down the payoffs in matrix form.

(a) If we add 10 to Player 1's payoff in every cell of the matrix, the set of pure Nash equilibria is unchanged.

True. Player 1's preference ranking over outcomes is unchanged if a constant amount is added to the payoff of every outcome.

(b) If we add 10 to Player 1's payoff in every cell of the matrix, the set of mixed Nash equilibria is unchanged.

True. With mixed strategies are allowed, the objects that players have preferences over are now probability distributions over payoffs. Adding 10 to the payoff of every outcome increases the expected value of any mixed strategy by 10, but does not change their relative rankings.

(c) Suppose one of Player 1's actions is weakly dominated. If we delete this row from the matrix, the matrix that remains has the same set of pure Nash equilibria as the original matrix.

False. A weakly dominated action may be played in a Nash equilibrium (for example, the voting example in Homework 1), so deleting this row may eliminate a NE. A weakly dominated action will not be played in a strict Nash equilibrium.

(d) Suppose one of Player 1's actions is strictly dominated. If we delete this row from the matrix, the matrix that remains has the same set of pure Nash equilibria as the original matrix.

True. A strictly dominated action is never played in Nash equilibria, so eliminating this row does not affect the set of Nash equilibria.

(e) For any action a_1 of Player 1, there exists at least one action for Player 2 that is a best response to a_1.

True. There is always a best response when choosing from a *finite* set of actions; if there are an *infinite* number of actions, then a best response may not exist.

Q2. **(30 pts)** Consider the following game where mixed strategies are allowed. Assume $a > 0$.

(a) **(15 pts)** What is the range of values of a for which the pure strategy T is strictly dominated?

	L	R
T	a,3	a,1
M	2,1	0,0
B	0,0	1,2

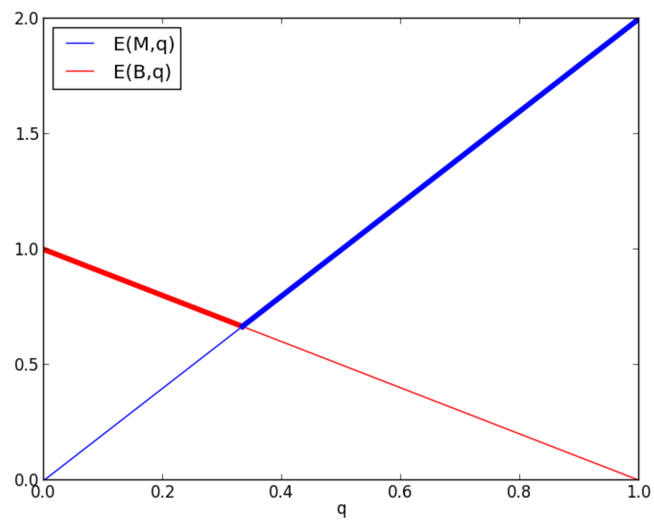
Let q be the probability that Player 2 plays L. Player 1's expected payoffs to each of his actions are:

$$E_1(T,q) = q(a) + (1-q)(a) = a$$

$$E_1(M,q) = q(2) + (1-q)(0) = 2q$$

$$E_1(B,q) = q(0) + (1-q)(1) = 1-q$$

T is strictly dominated if it is never a best response; that is, for all possible values of q between 0 and 1, either $E_1(M,q)$ or $E_1(B,q)$ is greater than $E_1(T,q)$. Below, we plot the values of $E_1(M,q)$ and $E_1(B,q)$ as a function of q:



The thick lines show the maximum of $E_1(M,q)$ and $E_1(B,q)$; if a is below the lowest point on these lines, which is at $q = 1/3$ and a payoff of $2/3$, then $E_1(T,q)$ is never the best response. This holds if $a < 2/3$.

(b) **(15 pts)** Assume that T is strictly dominated. Find the set of all pure and mixed Nash equilibria.

A strictly dominated action is played with zero probability in any Nash equilibrium. So we can eliminate the T row, and solve the remaining matrix The pure Nash equilibria are

	L	R
M	2,1	0,0
B	0,0	1,2

(M,L) and (B,R).

Let p be the probability that Player 1 plays M, and let q be the probability that Player 2 plays L. Player 1 is indifferent when $2q = 1 - q \rightarrow q = \frac{1}{3}$. For Player 2, taking p as given, his expected payoffs to his actions are:

$$E_2(L, p) = p(1) + (1 - p)0 = p$$

$$E_2(R, p) = p(0) + (1 - p)2 = 2 - 2p$$

Player 2 is indifferent when $p = 2 - 2p \rightarrow p = \frac{2}{3}$. The best response correspondences of each player are:

$$B_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{3} \\ 0 \leq p \leq 1 & \text{if } q = \frac{1}{3} \\ p = 1 & \text{if } q \ \frac{1}{3} \end{cases}$$

$$B_2(p) = \begin{cases} q = 0 & \text{if } p < \frac{2}{3} \\ 0 \leq q \leq 1 & \text{if } p = \frac{2}{3} \\ q = 1 & \text{if } p \ \frac{2}{3} \end{cases}$$

There is one additional mixed NE at $(p, q) = (\frac{2}{3}, \frac{1}{3})$.

Q3. **(30 pts)** A buyer and a seller of an object are trying to agree on the terms of a trade. The buyer and the seller simultaneously submit prices p_b, p_s, respectively. If $p_b \geq p_s$, trade occurs: the buyer pays p_b, the seller receives p_s, and the difference $p_b - p_s$ goes to a charity. If trade occurs, the buyer's payoff is his valuation of the object, v, minus the price he pays. The seller's payoff is the price received minus his cost, c. Assume $v > c \geq 0$. If trade does not occur, both players receive a payoff of 0.

(a) **(10 pts)** Formulate this situation as a strategic form game.

- Players: *Buyer*, *Seller*
- Actions: For both players, the action set is $[0, \infty)$, the set of non-negative real numbers.
- Payoff functions:

$$u_b(p_b, p_s) = \begin{cases} 0 & \text{if } p_b < p_s \\ v - p_b & \text{if } p_b \geq p_s \end{cases}$$

$$u_s(p_b, p_s) = \begin{cases} 0 & \text{if } p_b < p_s \\ p_s - c & \text{if } p_b \geq p_s \end{cases}$$

(b) **(10 pts)** Show that any (p_b, p_s) such that $p_b = p_s$ and $c \leq p_b \leq v$ is a Nash equilibrium.

Suppose $p_s = p_b$ is between c and v. Buyer's payoff is $v - p_s \geq 0$. If the buyer deviates and chooses $p_b < p_s$, he gets 0. If he chooses $p_b > p_s$, he gets $v - p_b$, which is less than $v - p_s$. Therefore, the buyer has no incentive to deviate.

Seller's payoff is $p_b - c \geq 0$. If the seller deviates and chooses $p_s < p_b$, he gets $p_s - c$ which is less than $p_b - c$. If he chooses $p_s > p_b$, he gets 0. Therefore, the seller has no incentive to deviate.

(c) **(10 pts)** Find the set of all pure strategy Nash equilibria.

The best response correspondences of the players are:

$$B_b(p_s) = \begin{cases} p_b = p_s & \text{if } p_s < v \\ 0 \leq p_b \leq p_s & \text{if } p_s = v \\ 0 \leq p_b < p_s & \text{if } p_s > v \end{cases}$$

$$B_s(p_b) = \begin{cases} p_s > p_b & \text{if } p_s < c \\ p_s \geq p_b & \text{if } p_s = c \\ p_s = p_b & \text{if } p_b > c \end{cases}$$

In addition to the set of (p_b, p_s) that satisfies $p_b = p_s$ and $c \leq p_b \leq v$, any (p_b, p_s) that satisfies $p_b \leq c$ and $p_s \geq v$ is a Nash equilibrium.

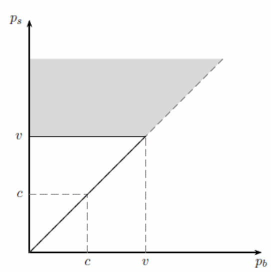


Figure 1: Buyer's best response correspondence

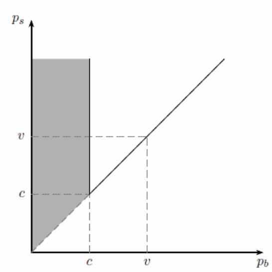


Figure 2: Seller's best response correspondence

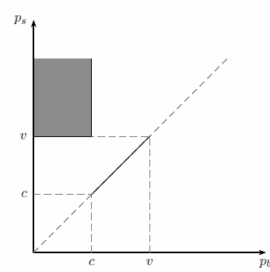


Figure 3: The set of Nash equilibria

Q4. **(30 pts)** Consider this model of a market that combines features of the Hotelling and Bertrand models. Buyers of a good are uniformly located on a line, starting at position 0 at the left and ending at position 1 on the right. There are two stores, A and B, located at position $\frac{1}{4}$ and 1, respectively. Each store charges its own price p_A and p_B. A buyer must choose which store to buy from. The total cost to the buyer of going to a store is the distance to the store's position, plus the price charged by the store. Buyers always choose the store with the lowest total cost.

(a) **(5 pts)** Suppose a buyer is at position x, where $0 \leq x \leq 1$. Write down the buyer's total cost if he chooses store A, and if he chooses store B.

The cost of buying from store A is:

$$\begin{cases} x - \frac{1}{4} + p_A & \text{if } x \geq \frac{1}{4} \\ \frac{1}{4} - x + p_A & \text{if } x < \frac{1}{4} \end{cases}$$

The cost of buying from store B is:

$$1 - x + p_B$$

(b) **(5 pts)** At what position is the buyer indifferent between A and B?

If $p_B + \frac{3}{4} = p_A$, all buyers from 0 to $\frac{1}{4}$ are indifferent. If $p_B + \frac{3}{4} < p_A$, all buyers will choose store B. If $p_B + \frac{3}{4} > p_A$, the indifference position x^* is where

$$x^* - \frac{1}{4} + p_A = 1 - x^* + p_B \Rightarrow x^* = \frac{p_B - p_A}{2} + \frac{5}{8}$$

(c) **(10 pts)** The length of the line that chooses store A is the quantity demanded from A (similarly for B). Write down the demand for A and B, as a function of p_A, p_B.

Assuming that both stores equally split regions where buyers are indifferent:

$$D_A(p_A, p_B) = \begin{cases} 0 & \text{if } p_B + \frac{3}{4} < p_A \\ \frac{1}{8} & \text{if } p_B + \frac{3}{4} = p_A \\ \frac{p_B - p_A}{2} + \frac{5}{8} & \text{if } p_B + \frac{3}{4} > p_A \end{cases}$$

$$D_B(p_A, p_B) = \begin{cases} 1 & \text{if } p_B + \frac{3}{4} < p_A \\ \frac{7}{8} & \text{if } p_B + \frac{3}{4} = p_A \\ \frac{3}{8} - \frac{p_B - p_A}{2} & \text{if } p_B + \frac{3}{4} > p_A \end{cases}$$

(d) **(10 pts)** Assume that costs are zero for both stores, and they each choose their own price to maximize profits. Find the set of pure Nash equilibrium (if any).

Each store's profit is price times demand. It cannot be a NE if A has zero demand or $\frac{1}{8}$ demand, because it can lower its price by an infinitesimally small amount and gain a large increase in demand (and therefore profits). Then, A's profit is $p_A(\frac{p_B - p_A}{2} + \frac{5}{8})$, B's profit is $p_B(1 - x^* = \frac{5}{8} - \frac{p_B - p_A}{2})$. Taking the derivative with respect to price and setting it to zero gives the best response function for each store:

$$B_A(p_B) = \frac{5}{8} + \frac{p_B}{2}$$

$$B_B(p_A) = \frac{3}{8} + \frac{p_A}{2}$$

The Nash equilibrium is at $(p_A = \frac{13}{12}, p_B = \frac{11}{12})$.