CUR 412: Game Theory and its Applications

Midterm Exam

Ronaldo Carpio October 24, 2014

Instructions:

- Please write your name in English.
- This exam is closed-book.
- Total time: 90 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(10 pts)** For the following questions, give an example, in matrix form, of a 2-player, 2-action game.

(a) **(5 pts)** Give an example of a game with no pure strategy Nash equilibria.

An example of a game with no pure strategy Nash equilibria is Matching Pennies:

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

(b) **(5 pts)** Give an example of a symmetric game with exactly one pure strategy Nash equilibrium.

A symmetric game is one where both players have the same actions, and the payoff matrix is of the form

	L	R
L	w,w	x,y
R	y,x	z,z

Prisoner's Dilemma is an example of a symmetric game with exactly one pure strategy Nash equilibrium.

Q2. **(30 pts)**

	L	C	R
T	4,2	3,0	1,1
M	1,2	2,4	0,3
B	1,1	4,2	2,4

(a) **(15 pts)** Find the set of pure strategy Nash equilibria.

The pure strategy NE are (T, L) and (B, R).

(b) **(15 pts)** Find the set of mixed strategy Nash equilibria.

First, notice that M is a strictly dominated action for Player 1; therefore, it will not be played with positive probability in any Nash equilibrium, and we can eliminate it. Once we eliminate it, notice that C is also strictly dominated for Player 2 and can be eliminated. We are left with the game:

	L	R
T	4,2	1,1
B	1,1	2,4

Let p be Player 1's probability on T, and q be Player 2's probability on L. For Player 1, the expected payoffs to T and B must be the same:

$$q \cdot 4 + (1 - q) \cdot 1 = q \cdot 1 + (1 - q) \cdot 2$$

which gives $q = 1/4$. For Player 2, the expected payoffs to L and R must be the same:

$$p \cdot 2 + (1 - p) \cdot 1 = p \cdot 1 + (1 - p) \cdot 4$$

which gives $p = 3/4$. The only non-pure strategy Nash equilibrium is $((3/4, 1/4), (1/4, 3/4))$.

Q3. **(30 pts)** Three firms are considering whether to enter a new market. A firm that does not enter gets a payoff of 0. A firm that does choose to enter must pay a cost of 62 and makes revenues of $\frac{150}{n}$, where n is the total number of firms that choose to enter; its payoff is the amount of profits it makes.

(a) **(15 pts)** Find the set of pure strategy Nash equilibria.

Let the action set be $\{In, Out\}$. If all three players choose Out, they each get a payoff of 0. If exactly one firm chooses In, it gets a payoff of 88, while the other two get 0. If exactly two firms choose In, they both get a payoff of 13, while the other firm gets 0. If all three firms choose In, they all get a payoff of -12.

- Suppose all three players choose Out. Any firm can increase its payoff from 0 to 88 by switching to In, so this is not a NE.
- Suppose exactly one player chooses In. One of the firms that chose Out can increase its payoff from 0 to 13 by switching to In, so this is not a NE.
- Suppose exactly two players choose In. The remaining firm that chose Out will decrease its payoff from 0 to -12 by switching to In. Each firm that chose In will decrease its payoff from 13 to 0 by switching to Out. Therefore, this is a NE.
- Suppose all three players choose In. Each firm can increase its payoff from -12 to 0 by switching to Out, so this is not a NE.

Therefore, there are three pure strategy NE, where firm $i = \{1,2,3\}$ chooses Out, and the remaining two firms choose In.

(b) **(15 pts)** Find the symmetric mixed strategy Nash equilibrium, where all three firms enter with the same probability.

Let p be the probability of choosing In. Suppose we are firm 1; the probability that $n = 1$ (if both other firms choose Out) is $(1-p)^2$. The probability that $n = 2$ (if exactly one other firm chooses In) is $2p(1-p)$. The probability that $n = 3$ (if both other firms choose In) is p^2. The expected payoffs to In and Out must be the same:

$$EV(In) = (1-p)^2 88 + 2p(1-p)13 + p^2(-12)$$

$$EV(Out) = 0$$

$EV(In) = EV(Out)$ when $p = 4/5$.

Q4. **(30 pts)** Two firms are competing in a market and their products are imperfect substitutes for one another. The demand functions for the products of firm 1 and 2 are given by:

$$q_1(p_1,p_2) = \begin{cases} a - bp_1 + p_2 & \text{if } a - bp_1 + p_2 \geq 0 \\ 0 & \text{if } a - bp_1 + p_2 < 0 \end{cases}$$

$$q_2(p_1,p_2) = \begin{cases} a - bp_2 + p_1 & \text{if } a - bp_2 + p_1 \geq 0 \\ 0 & \text{if } a - bp_2 + p_1 < 0 \end{cases}$$

where p_1 and p_2 are the prices charged by firm 1 and firm 2, respectively. Prices must be non-negative, and assume that $a > 0$ and $b > 1$. Assume that costs are zero, so profit for firm i is $p_i q_i(p_1,p_2)$.

(a) **(15 pts)** Formulate this situation as a strategic form game.

- Players: The two firms.
- Actions: The set of non-negative numbers, $[0,\infty)$.
- Payoffs:

$$u_1(p_1,p_2) = \begin{cases} p_1(a - bp_1 + p_2) & \text{if } a - bp_1 + p_2 \geq 0 \\ 0 & \text{if } a - bp_1 + p_2 < 0 \end{cases}$$

$$u_2(p_1,p_2) = \begin{cases} p_2(a - bp_2 + p_1) & \text{if } a - bp_2 + p_1 \geq 0 \\ 0 & \text{if } a - bp_2 + p_1 < 0 \end{cases}$$

(b) **(15 pts)** Find the set of all pure strategy Nash equilibria.

We will find the best response function. Suppose we are firm 1 and p_2 is given. The best response of firm 1 cannot exceed $p_1 > \frac{a+p_2}{b}$, since in that case $q_1(p_1,p_2) = 0$; firm 1 can choose a slightly lower positive price and still make a positive profit. Firm 1 will maximize $p_1(a - bp_1 + p_2)$, as long as the solution does not exceed $p_1 > \frac{a+p_2}{b}$. We set the derivative to 0:

$$-bp_1 + a - bp_1 + p_2 = 0$$

which gives $p_1 = \frac{a+p_2}{2b}$. We want to check that the payoff with this solution is always positive, which is true:

$$\left(a - b\frac{a+p_2}{2b} + p_2\right)\frac{a+p_2}{2b} = \frac{(a+p_2)^2}{4b} > 0$$

Therefore, the best response function of firm 1 is $B_1(p_2) = \frac{a+p_2}{2b}$. Symmetrically, the best response function of firm 2 is $B_2(p_1) = \frac{a+p_1}{2b}$. Combining these two gives the solution $p_1 = p_2 = \frac{a}{2b-1}$.