CUR 412: Game Theory and its Applications

Midterm Exam

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Instructions:

- Please write your name in English.
- This exam is closed-book.
- Total time: 90 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(10 pts)** For the following questions, give an example, in matrix form, of a 2-player, 2-action game.

(a) **(5 pts)** Give an example of a game where the NE are not Pareto-optimal (that is, there is a non-NE outcome in which both players are better off).

	C	D
C	2,2	0,3
D	3,0	1,1

The Prisoner's Dilemma is an example of such a game. The outcome (C, C) results in a higher payoff for both players than (D, D), which is the only pure NE.

(b) **(5 pts)** Give an example of a symmetric game where both players prefer to play different actions from each other.

	A	B
A	0,0	1,1
B	1,1	0,0

In a game such as this one, both players must receive a higher payoff from the outcomes (A, B) and (B, A) than the outcomes (A, A) and (B, B).

Q2. **(30 pts)**

	L	R
T	3,2	1,1
M	2,0	0,0
B	1,1	2,3

(a) **(15 pts)** Find all dominated actions. State if it is strictly or weakly dominated.

For Player 1, M is strictly dominated by T. There are no other dominated actions.

(b) **(15 pts)** Find all pure and mixed Nash equilibria.

We can eliminate M since it is strictly dominated. We are left with the 2×2 game

	L	R
T	3,2	1,1
B	1,1	2,3

The pure NE are (T, L) and (B, R). Let p, q be the probability that T, L are played by Player 1, Player 2 respectively. The expected payoffs for each action of Player 1 are:

- $E_1(T) = 3q + 1(1-q) = 1 + 2q$
- $E_1(B) = 1q + 2(1-q) = 2 - q$

Setting $E_1(T) = E_1(B)$ gives $q = 1/3$. For player 2, the expected payoffs are:

- $E_2(L) = 2p + 1(1-p) = 1 + p$
- $E_2(R) = 1p + 3(1-p) = 3 - 2p$

Setting $E_2(L) = E_2(R)$ gives $p = 2/3$. Therefore, there is one mixed NE: $\{(2/3, 1/3), (1/3, 2/3)\}$.

Q3. **(30 pts)** A citizen (player 1) must choose whether to file taxes honestly, or to cheat. The tax collector (player 2) decides how much effort to invest in auditing; he chooses a real number a between 0 and 1, which costs the tax collector $100a^2$. If the citizen is honest, he gets a payoff of 0 and the tax collector gets a payoff of $-100a^2$. If the citizen cheats, then a is the probability of getting caught. If the citizen cheats, the payoffs are:

- If the citizen is caught (with probability a), he gets a payoff of -100 and the tax collector gets a payoff of $100 - 100a^2$.
- If the citizen is not caught (with probability $1-a$), he gets a payoff of 50 and the tax collector gets a payoff of $-100a^2$.

(a) **(5 pts)** Suppose the tax collector believes the citizen is honest with probability 1. Find the best response level of a.

The tax collector's expected payoff as a function of a is $-100a^2$, which is maximized at $a = 0$.

(b) **(5 pts)** Suppose the tax collector believes the citizen is cheating with probability 1. Find the best response level of a.

The tax collector's expected payoff as a function of a is $a(100 - 100a^2) + (1-a)(-100a^2)$. The derivative with respect to a is $100 - 200a$; therefore, the function is maximized when $a = 1/2$.

(c) **(10 pts)** Suppose the tax collector believes the citizen is cheating with probability p. Find the best response level of a as a function of p.

The tax collector's expected payoff as a function of a is $p(a(100-100a^2)+(1-a)(-100a^2))+(1-p)(-100a^2)$. The derivative with respect to a is $100(p-2a)$; therefore, the function is maximized when $a = p/2$.

(d) **(10 pts)** Find all pure strategy NE, or show that none exists.

At a NE, both players must play best responses to the other. Let's try each of Player 1's actions, find the best response by Player 2, and then check if the best response by Player 1 is the same as our original assumption.

Suppose Player 2 chooses a. Player 1's expected payoffs to his actions are:

$$E_1(Honest) = 0, \qquad E_1(Cheat) = a(-100) + (1-a)(50) = 50 - 150a$$

$Honest$ is the best response if $0 \geq 50 - 150a \Rightarrow a \geq 1/3$; $Cheat$ is the best response if $a \leq 1/3$.

- Suppose Player 1 chooses $Honest$; then Player 2's best response is $a = 0$, and Player 1's best response to $a = 0$ is $Cheat$, which is inconsistent.
- Suppose Player 1 chooses $Cheat$; then Player 2's best response is $a = 1/2$, and Player 1's best response to $a = 1/2$ is $Honest$, which is inconsistent.

Therefore, there is no pure NE.

Q4. **(30 pts)** Consider the Bertrand model of oligopoly. Suppose there are two firms with the same unit cost c; each firm chooses a price p_1, p_2. Buyers only buy from the firm offering the lowest price; if both firms offer the same price, then market demand is split equally. Market demand at price p is given by $D(p) = \alpha - 2p$ for $2p \leq \alpha$, $D(p) = 0$ for $2p > \alpha$. Assume $c < \alpha/2$.

(a) **(10 pts)** Find the Nash equilibrium prices, output levels, and profits for each firm.

For firm i, the profit per unit is $(p_i - c)$. The payoff function is:

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c)(\alpha - 2p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - 2p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

The only NE is when both firms choose $p_1 = p_2 = c$, which results in an output level of $\alpha - 2c$ and zero profits for both firms.

(b) **(10 pts)** Suppose both firms combined to form a monopoly. Find the equilibrium price, output level, and profit.

For a monopoly, profit is given by $(p-c)(\alpha - 2p)$, which is maximized at $p = \frac{\alpha+2c}{4}$. Output is $\frac{\alpha-2c}{2}$ and profit is $\frac{(\alpha-2c)^2}{8}$.

(c) **(10 pts)** Now, suppose that both firms engage in *price matching*: if a customer can find a lower price elsewhere, the firm will reduce their price to match it. Assume that if firms choose (p_1, p_2), the price that both firms end up facing is $\min(p_1, p_2)$. Write down the payoff function, and show that it is a NE when both firms charge the monopoly price.

The payoff function becomes:

$$\pi_i(p_i, p_j) = \begin{cases} \frac{1}{2}(p_i - c)(\alpha - 2p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - 2p_i) & \text{if } p_i = p_j \\ \frac{1}{2}(p_j - c)(\alpha - 2p_j) & \text{if } p_i > p_j \end{cases}$$

Suppose $p_1 = p_2 = \frac{\alpha+2c}{4}$, the monopoly price. If one firm deviates by increasing its price, there is no effect, since the minimum of p_1, p_2 remains the same. If the firm deviates by decreasing its price, profit for both firms must fall, because part (b) established that the monopoly price maximizes the function $(p - c)(\alpha - 2p)$. Therefore, this is a NE. Each firm's output is half of the monopoly quantity: $\frac{\alpha-2c}{4}$. Profit is half of the monopoly profit: $\frac{(\alpha-2c)^2}{16}$.