CUR 412: Game Theory and its Applications

Midterm Exam

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Instructions:

- Please write your name in English.
- This exam is closed-book.
- Total time: 90 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(10 pts)** For the following questions, give an example, in matrix form, of a 2-player, 2-action game.

(a) **(5 pts)** Give an example of a game with a (strictly) dominant strategy equilibrium.

For a 2-player, 2-action game to have a dominant strategy equilibrium, each player must have a strictly dominant strategy. One such game is the Prisoner's Dilemma:

	C	D
C	2,2	0,3
D	3,0	1,1

For both players, D is a strictly dominant strategy.

(b) **(5 pts)** Give an example of a symmetric game with a Nash equilibrium that is not symmetric.

	L	R
T	0,0	1,1
B	1,1	0,0

This game is symmetric, since the payoffs to each player in each outcome are the same if Player 1 and Player 2 are switched. The NE are (T, R) and (B, L), but they are not symmetric equilibria, in which both players choose the same action.

Q2. **(25 pts)**

	L	C	R
T	3,2	4,0	1,1
M	2,0	3,3	0,0
B	1,1	0,2	2,3

(a) **(10 pts)** Find the set of outcomes that remain after iterated elimination of strictly dominated strategies.

First, M is strictly dominated by T, so it can be eliminated. Then, C is strictly dominated by R, so it can be eliminated. The outcomes that remain are

	L	R
T	3,2	1,1
B	1,1	2,3

(b) **(15 pts)** Find all pure and mixed Nash equilibria.

The two pure strategy NE are (T, L) and (B, R).

Let $p, 1-p$ be the probabilities placed on T, B respectively, and let $q, 1-q$ be the probabilities placed on L, R respectively. Player 1's expected payoffs to his actions are:

$$E_1(T) = q \cdot 3 + (1-q) \cdot 1 = 1 + 2q$$

$$E_2(B) = q \cdot 1 + (1-q) \cdot 2 = 2 - q$$

Solving $1 + 2q = 2 - q$ gives $q = \frac{1}{3}$. Player 2's expected payoffs to his actions are:

$$E_2(L) = p \cdot 2 + (1-p) \cdot 1 = 1 + p$$

$$E_2(R) = p \cdot 1 + (1-p) \cdot 3 = 3 - 2p$$

Solving $1 + p = 3 - 2p$ gives $p = \frac{2}{3}$. Therefore, there is one mixed NE at $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$.

Q3. **(25 pts)** Consider the Bertrand oligopoly game with two firms. Firm 1's cost function is $c_1(q_1) = q_1$, and Firm 2's cost function is $c_2(q_2) = 2q_2$. Total demand is given by $P = 100 - Q$, where $Q = q_1 + q_2$. Firms choose prices p_1, p_2 respectively, where $p_1, p_2 \geq 0$, and prices must be an integer multiple of 0.01 (so 0.01, 0.02, 0.03, ... are valid prices). Customers only buy from the firm offering the lowest price; if there is a tie, the market is split equally.

(a) **(5 pts)** Write down the profit functions of both firms.

$$\pi_1(p_1, p_2) = \begin{cases} (100 - p_1)(p_1 - 1) & \text{if } p_1 < p_2 \\ \frac{1}{2}(100 - p_1)(p_1 - 1) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

$$\pi_2(p_1, p_2) = \begin{cases} (100 - p_2)(p_2 - 2) & \text{if } p_2 < p_1 \\ \frac{1}{2}(100 - p_2)(p_2 - 2) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_2 > p_1 \end{cases}$$

(b) **(20 pts)** Find the set of Nash equilibria.

First, let's find the prices that maximize the quadratics $(100 - p_1)(p_1 - 1)$ and $(100 - p_2)(p_2 - 2)$.

$$\frac{\partial \pi_1}{\partial p_1} = 101 - 2p, \frac{\partial \pi_2}{\partial p_2} = 102 - 2p$$

By setting the derivative to 0, we get $p_1^* = 50.5$ and $p_2^* = 51$. This is far higher than the marginal costs of each firm, which are 1 and 2, respectively, so we know that if prices are around 1 and 2, firms have an incentive to raise their price if they can.

Next, each firm will not sell below marginal cost in a NE: if $p_1 \leq p_2$, then p_1 must be ≥ 1.00, and if $p_2 \leq p_1$, then p_2 must be ≥ 2.00, otherwise one firm would make a negative profit, and that firm would have an incentive to deviate by raising their price and getting zero customers.

In any NE, Firm 2 will get zero customers, since if it offered a p_2 that did not make a loss (i.e. $p_2 \geq 2.00$), Firm 1 can always undercut it while making a positive profit. We can also say that in any NE, if Firm 1 is getting nonzero customers, then it must be that $p_1 + 0.01 = p_2$, since Firm 1 has an incentive to raise its price as high as possible while remaining below p_2. This tells us that the set of Nash equilibria will contain $(p_1, p_2) = (1.01, 1.02), ..., (2.00, 2.01)$. In each of these outcomes:

- Firm 1 will make a smaller profit if it lowers p_1, and its profits will be divided in half if it raises p_1 by 0.01, or go to zero if it raises p_1 by more than 0.01.
- Firm 2 is making zero profits. If it raises p_2, it will also make zero profits. If it lowers p_2, it will make either a negative profit (if $p_2 < 2.00$) or zero profit (if $p_2 = 2.00$).

Neither firm has an incentive to deviate, so these are NE.

Finally, we can eliminate the cases where $2.01 \leq p_1 < p_2$, since Firm 2 can undercut or tie p_1 while still making a positive profit. So the set of NE are $(1.01, 1.02), ..., (2.00, 2.01)$.

Q4. **(40 pts)** Consider a sealed-bid, *all-pay* auction (that is, all bidders pay the same cost, whether they win or not). This can be used to model situations such as government lobbying, where all bidders have to spend resources in order to have a chance at winning. Suppose there are two bidders with valuations v_1, v_2 respectively, and assume that $v_1 > v_2 > 0$. Let x_1, x_2 denote each player's bid, where $x_1 \geq 0, x_2 \geq 0$. If there is a tie, assume Player 1 wins.

Suppose the auction is *second-price* (the highest bidder wins, and both players pay the second highest bid).

(a) **(5 pts)** Write down the payoff functions for both players.

$$u_1(x_1, x_2) = \begin{cases} v_1 - x_2 & \text{if } x_1 \geq x_2 \\ -x_1 & \text{if } x_1 < x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} v_2 - x_1 & \text{if } x_2 > x_1 \\ -x_2 & \text{if } x_2 \leq x_1 \end{cases}$$

(b) **(15 pts)** Find the set of Nash equilibria.

This game has the same structure as War of Attrition. Player 1's best response to x_2 is:

- If $x_2 \leq v_1$, then any $x_1 \geq x_2$ is a best response, since Player 1 will win the auction, and the cost will be determined by x_2 (and therefore unaffected by the particular choice of x_1).
- If $x_2 > v_1$, then $x_1 = 0$ is the best response, since Player 1 will lose the auction, and the cost is increasing in x_1.

Player 2's best response to x_1 is:

- If $x_1 < v_2$, then any $x_2 > x_1$ is a best response, since Player 2 will win the auction, and the cost will be determined by x_1 (and therefore unaffected by the particular choice of x_2).
- If $x_1 \geq v_2$, then $x_2 = 0$ is the best response, since Player 2 will lose the auction, and the cost will be increasing in x_2.

The set of NE are: $(x_1 = 0, x_2 \geq v_1)$ and $(x_1 \geq v_2, x_2 = 0)$.

Now, suppose the auction is *first-price* (the highest bidder wins, and both players pay the highest bid).

(c) **(5 pts)** Write down the payoff functions for both players.

$$u_1(x_1, x_2) = \begin{cases} v_1 - x_1 & \text{if } x_1 \geq x_2 \\ -x_2 & \text{if } x_1 < x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} v_2 - x_2 & \text{if } x_2 > x_1 \\ -x_1 & \text{if } x_2 \leq x_1 \end{cases}$$

(d) **(15 pts)** Find the set of Nash equilibria.

We will show that there are no NE.

- Suppose that $x_1 \neq x_2$. The winner's payoff is increasing in his own bid, so he always has an incentive to lower his bid by a small amount ϵ, while still remaining above the losing bid.
- Suppose that $x_1 = x_2$, so Player 1 wins. Player 2 has an incentive to increase his bid by a small amount ϵ, since he will become the winner and his payoff will increase by $v_2 - \epsilon$.

Therefore, some player always has an incentive to deviate, and there is no NE.