CUR 412: Game Theory and its Applications

Midterm Solutions

Q1. **(10 pts)**. Answer whether each of the following statements is True or False. You don't have to provide an explanation.

(a) If a game has a finite number of players and actions, there always exists a pure strategy Nash equilibrium.

False. For example, Matching Pennies has no pure strategy Nash equilibrium. It has been proven, however, that a *mixed* strategy Nash equilibrium always exists in a game with finite players and actions (see Proposition 119.1 in the textbook).

(b) If two different actions are played (with positive probability) in a mixed strategy Nash equilibrium, then these actions give the same expected payoffs (given that the other players do not deviate from the equilibrium).

True. This is Proposition 116.2 in the textbook.

(c) In a 2-player strategic game with ordinal preferences, for any action a_i of player i, there exists at least one action for player j that is a best response to a_i.

False. A best response may not exist for all values of a_i (for example, in Bertrand's duopoly game).

(d) If an action is not strictly dominated in a strategic game without randomization, it cannot be strictly dominated when randomization is allowed.

False. See Figure 120.1 in the textbook.

Q2. **(30 pts)** Consider the following game:

	L	R
T	4,4	0,2
M	2,0	2,2
B	3,0	1,0

(a) **(15 pts)** Find the set of pure strategy Nash equilibria.

(T, L) and (M, R)

(b) **(15 pts)** Find a mixed strategy Nash equilibrium in which Player 1 plays all three actions with positive probability.

Let q be Player 2's probability of playing L. The expected payoffs to each of Player 1's actions are:

$$E_1(T) = 4q$$

$$E_1(M) = 2$$

$$E_1(B) = 3q + 1(1 - q) = 2q + 1$$

In a mixed strategy NE where all actions are played with positive probability, these expected values must be equal to each other. Solving $4q = 2 = 2q + 1$ gives $q = \frac{1}{2}$. Likewise, let $p_1, p_2, 1 - p_1 - p_2$ be the probabilities that Player 1 places on T, M, B respectively. We know that Player 2 places positive probability on both of his actions, so expected payoffs must be equal:

$$E_2(L) = 4p_1$$

$$E_2(R) = 2p_1 + 2p_2$$

$4p_1 = 2p_1 + 2p_2$ gives us $p_1 = p_2$. Therefore, a mixed strategy profile is a MSNE if it satisfies this form with $0 < p < \frac{1}{2}$:

$$\left((p, p, 1 - 2p), (\frac{1}{2}, \frac{1}{2})\right)$$

Q3. **(30 pts)** Two individuals, player 1 and player 2, choose how much effort to put into a joint project. If player 1 puts in effort level $x \geq 0$ and player 2 puts in effort $y \geq 0$, their payoffs are given by:

$$u_1(x, y) = (x + y + xy) - 2x^2$$

$$u_2(x, y) = v(x + y + xy) - 2y^2$$

where v is a constant, $0 \leq v \leq 2$.

(a) **(20 pts)** Find the set of Nash equilibria of this game as a function of v.

Player 1's problem is to maximize $(x + y + xy) - 2x^2$. If $x > 0$, the first-order condition is:

$$\frac{\partial}{\partial x} = y - 4x + 1 = 0$$

which gives us Player 1's best response function $B_1(y) = \frac{1+y}{4}$. Likewise, Player 2's best response function is $B_2(x) = \frac{v(1+x)}{4}$. Solving these equations gives us

$$x = \frac{v + 4}{16 - v}, y = \frac{5v}{16 - v}$$

(b) **(10 pts)** Find an action profile that maximizes the sum of their payoffs.

The sum of the payoffs is

$$(v + 1)(x + y + xy) - 2x^2 - 2y^2$$

To find x, y that maximizes this function, we take the partial derivatives with respect to x, y and set to zero. This gives:

$$\frac{\partial}{\partial x} = (v + 1)(1 + y) - 4x = 0$$

$$\frac{\partial}{\partial y} = (v + 1)(1 + x) - 4y = 0$$

Note that the equations are symmetric, which implies $x = y$. We can then solve for x, y:

$$x = y = \frac{v+1}{3-v}$$

Q4. **(30 pts)** Two people are trying to share a cake of size 100. They each simultaneously announce a real number from 0 to 100. If the sum of the numbers is less than or equal to 100, each person receives the number he announced. If the sum is greater than 100, then the player who announced the smaller number, say x, receives x while the other person receives $100 - x$. If the sum is greater than 100 and both numbers are the same, then each receives 50.

(a) **(10 pts)** Formulate this situation as a strategic game.

- Players: the two people
- Actions: Each player's action is a real number $a_i, 0 \leq a_i \leq 100$
- Preferences: Each player's preferences can be represented by the following payoff functions:

$$u_1(a_1, a_2) = \begin{cases} a_1 & \text{if } a_1 + a_2 \leq 100 \\ a_1 & \text{if } a_1 + a_2 > 100, a_1 < a_2 \\ 100 - a_2 & \text{if } a_1 + a_2 > 100, a_1 > a_2 \\ 50 & \text{if } a_1 + a_2 > 100, a_1 = a_2 \end{cases}$$

$$u_2(a_1, a_2) = \begin{cases} a_2 & \text{if } a_1 + a_2 \leq 100 \\ a_2 & \text{if } a_1 + a_2 > 100, a_2 < a_1 \\ 100 - a_1 & \text{if } a_1 + a_2 > 100, a_2 > a_1 \\ 50 & \text{if } a_1 + a_2 > 100, a_1 = a_2 \end{cases}$$

(b) **(10 pts)** Show that $(50, 50)$ is a Nash equilibrium.

Suppose $a_2 = 50$.

- If Player 1 plays $a_1 = 50$, payoff is 50.
- If $a_1 < 50$, payoff is less than 50.
- If $a_1 > 50$, payoff is $100 - a_2 = 50$.

$a_1 = 50$ is a best response to $a_2 = 50$, and vice versa. Therefore, $(50, 50)$ is a Nash equilibrium.

(c) **(10 pts)** Is there any other Nash equilibrium? Prove your answer.

The best response correspondence of player i is:

$$B_i(a_j) = \begin{cases} \{a_i | 100 - a_j \leq a_i \leq 100\} & \text{if } a_j \leq 50 \\ \emptyset & \text{if } a_j > 50 \end{cases}$$

The only intersection of B_1 and B_2 is at $(50, 50)$.

Or, we can argue this way:

- Suppose $a_j > 50$. Then Player i can increase his payoff, no matter what a_i is, by choosing a number closer to a_j (but still less than a_j). Therefore, this cannot be a NE.
- Suppose $a_i < a_j \leq 50$. Player i can increase his payoff by choosing 50. Not a NE.
- Suppose $a_i = a_j < 50$. Either player can increase his payoff by choosing 50. Not a NE.

The only other case is $a_i = a_j = 50$, which therefore is the only NE.