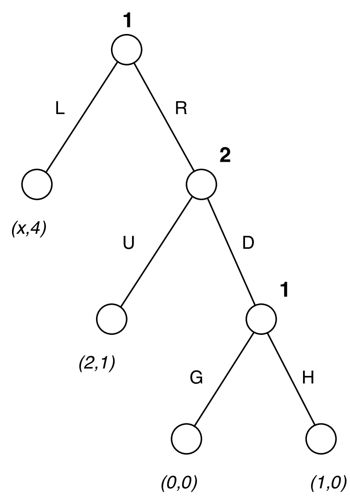
**CUR 412: Game Theory and its Applications**

**Solutions to Final Exam - 2013**

Q1. **(24 pts)** Consider the following extensive form game:




(a) **(8 pts)** Suppose $x = 1$. Find the set of pure strategy Nash equilibria and subgame perfect Nash equilibria.

Using backwards induction, we find that the unique SPNE is $(RH, U)$.

The strategic form of the game is given by this matrix:

| | $U$ | $D$ |
|---|---|---|
| $LG$ | 1,4 | 1,4 |
| $LH$ | 1,4 | 1,4 |
| $RG$ | 2,1 | 0,0 |
| $RH$ | 2,1 | 1,0 |

The set of NE are: $(LG, D), (LH, D), (RG, U), (RH, U)$.

(b) **(8 pts)** Find the range of $x$ for which $(R, U)$ is the unique subgame perfect NE outcome.

The unique SPNE of the subgame beginning after history $R$ is $(U, H)$ with a payoff of $(2, 1)$. Therefore, at the beginning of the game, $R$ is the unique optimal choice if $x < 2$.

(c) **(8 pts)** Find the range of $x$ for which $L$ is a Nash equilibrium outcome.

The matrix form of the strategic game with $x$ is:

| | $U$ | $D$ |
|---|---|---|
| $LG$ | x,4 | x,4 |
| $LH$ | x,4 | x,4 |
| $RG$ | 2,1 | 0,0 |
| $RH$ | 2,1 | 1,0 |

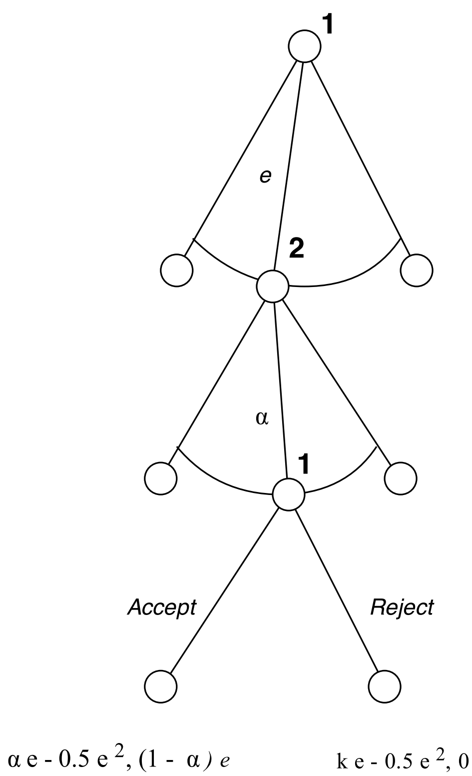
We want to find the range of $x$ that makes at least one of $(LG, U)$, $(LG, D)$, $(LH, U)$, $(LH, G)$ a NE; then $L$ is a NE outcome. If $x < 1$, then none of these are NE. Therefore, the range of $x$ is $x \geq 1$.

Q2. **(25 pts)** Suppose two people are working on a project and must decide how to split its value. Player 1 can exert an effort $e \geq 0$ with a cost $c(e) = 0.5e^2$. If Player 1 and Player 2 can agree on how to divide the project, then a total value of $v(e) = e$ is produced. If they cannot come to an agreement, the project produces less value, $y(e) = ke, 0 \leq k \leq 1$, that goes to Player 1 only. The game has three stages:

1. Player 1 chooses effort $e \geq 0$.
2. Player 2 observes $e$, chooses an offer $\alpha, 0 \leq \alpha \leq 1$.
3. Player 1 observes $\alpha$, chooses to *Accept* or *Reject*.

If Player 1 *Accept*s, his payoff is $\alpha e - 0.5e^2$ and Player 2's payoff is $(1 - \alpha)e$. If Player 1 *Reject*s, his payoff is $ke - 0.5e^2$ and Player 2's payoff is zero.

(a) **(5 pts)** Draw the tree representation of this game.




(b) **(10 pts)** Find Player 1's choice of $e$ in a subgame perfect NE.

In the last stage, taking $e, \alpha$ as given, *Accept* is at least as good as *Reject* if $\alpha e - 0.5e^2 \geq ke - 0.5e^2$, or if $\alpha \geq k$. There are two possible optimal strategies for Player 1:

(a) Accept only if $\alpha > k$, in which case there is no SPNE.

(b) Accept if $\alpha \geq k$, in which case there is a SPNE. We will assume Player 1 uses this strategy.

In the second stage, taking $e$ as given, Player 2's payoff is:

(a) If Player 2 offers $\alpha < k$, Player 1 will *Reject*, so Player 2 gets 0.

(b) If Player 2 offers $\alpha \geq k$, Player 1 will *Accept*, so Player 2 gets $(1 - \alpha)e$.

Since Player 2's payoff when Player 1 *Accept*s is decreasing in $\alpha$, the optimal choice for Player 2 is $\alpha = k$.

In the first stage, Player 1's payoff is:

(a) If Player 1 offers $e = 0$, he gets a payoff of 0.

(b) If Player 1 offers $e > 0$, Player 2 offers $\alpha = k$, then Player 1 *Accept*s, giving a payoff of $ke - 0.5e^2$. This is maximized w.r.t. $e$ at $e = k$, giving a payoff of $0.5k^2$.

Therefore, if $k > 0$, the optimal choice is $e = k$, while if $k = 0$, the optimal choice is $e = 0$. Combining, the optimal choice is $e = k$.

(c) **(10 pts)** Suppose Player 2 could choose $k, 0 \leq k \leq 1$ *before* the start of the game. What would he choose? What would be Player 1's choice of $e$ in SPNE?

Player 2's payoff in the SPNE is $(1 - k)k$, which is maximized at $k = \frac{1}{2}$. Player 1's choice of $e$ is also $\frac{1}{2}$.

Q3. **(24 pts.)** Consider the infinitely repeated version of the following game:

| | $H$ | $D$ |
|---|---|---|
| $H$ | 1,1 | 3,0 |
| $D$ | 0,3 | 2,2 |

The payoff of player $i$ to any infinite sequence of payoffs $\{u_{it}\}$ is given by the normalized discounted sum of payoffs:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_{it}$$

where $0 < \delta < 1$.

(a) **(12 pts.)** For what values of $\delta$, if any, does it constitute a subgame perfect equilibrium when both players choose this strategy?

- Choose $D$ in period 1.
- Choose $D$ after any history in which both players have always played $D$.
- Choose $H$ after any other history.

The behavior of a player who uses this strategy depends only on whether the history is all $(D, D)$ or not. Let's consider each type of history, and check if the strategy satisfies the one-shot deviation property. We'll consider only Player 1's problem, since the game is symmetric.

(a) Suppose the history is all $(D, D)$. If Player 1 does not deviate, the sequence of outcomes will be $(D, D), (D, D), ...$ with payoffs $2, 2, ...$ with discounted average of 2.

If Player 1 does a one-shot deviation, the sequence of outcomes will be $(H, D), (H, H), (H, H), ...$ with payoffs $3, 1, 1, ...$ with discounted average $3(1 - \delta) + \delta$. Not deviating is optimal if

$$2 \geq 3(1-\delta) + \delta \rightarrow \delta \geq \frac{1}{2}$$

(b) Suppose the history is *not* all $(D, D)$. If Player 1 does not deviate, the sequence of outcomes will be $(H, H), (H, H), ...$ with payoffs $1, 1, ...$ with discounted average of 1.

If Player 1 does a one-shot deviation, the sequence of outcomes will be $(D, H), (H, H), (H, H), ...$ with payoffs $0, 1, 1, ...$ with discounted average of $\delta$. This is always less than 1, therefore it is always optimal to not deviate.

Combining both conditions, this strategy can support a SPNE when $\delta \geq \frac{1}{2}$.

(b) **(12 pts.)** Suppose the game is modified to have the following payoffs:

| | $H$ | $D$ |
|---|---|---|
| $H$ | 0,0 | 3,1 |
| $D$ | 1,3 | 2,2 |

For what values of $\delta$, if any, does it constitute a subgame perfect equilibrium when both players choose the strategy in part (a)?

(a) Suppose the history is all $(D, D)$. If Player 1 does not deviate, the sequence of outcomes will be $(D, D), (D, D), ...$ with payoffs $2, 2, ...$ with discounted average of 2.

If Player 1 does a one-shot deviation, the sequence of outcomes will be $(H, D), (H, H), (H, H), ...$ with payoffs $3, 0, 0, ...$ with discounted average $3(1 - \delta)$. Not deviating is optimal if

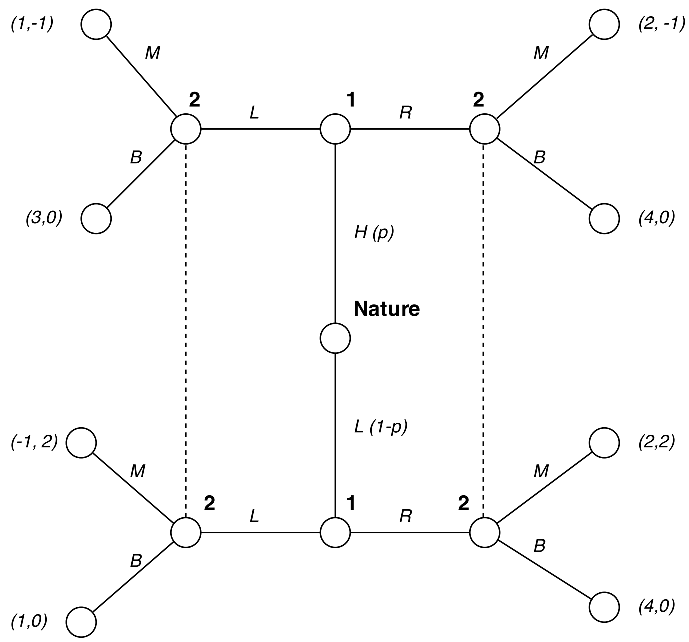
$$2 \geq 3(1 - \delta) \rightarrow \delta \geq \frac{1}{3}$$

(b) Suppose the history is *not* all $(D, D)$. If Player 1 does not deviate, the sequence of outcomes will be $(H, H), (H, H), ...$ with payoffs $0, 0, ...$ with discounted average of 0.

If Player 1 does a one-shot deviation, the sequence of outcomes will be $(D, H), (H, H), (H, H), ...$ with payoffs $1, 0, 0, ...$ with discounted average of $1 - \delta$. This is always greater than 0, therefore it is never optimal to not deviate.

Combining both conditions, this strategy is not a SPNE for any $0 < \delta < 1$.

Q4. **(27 pts.)** Consider this signaling game. Nature chooses $H, L$ with probability $p = \frac{1}{2}$.




(a) **(3 pts)** For Player 1 and Player 2, list the histories in each player's information sets.

Player 1's information sets are: $H, L$. Player 2's information sets are: $\{HL, LL\}$ and $\{HR, LR\}$.

(b) **(4 pts)** For each of Player 1 and Player 2's information sets, list their pure strategies.

Player 1's pure strategies are: $LL, LR, RL, RR$, where the first letter is the action if $H$ occurs and the second letter is the action if $L$ occurs. Player 2's pure strategies are: $(M|L, M|R), (M|L, B|R), (B|L, M|R), (B|L, B|R)$ where $(M|L, M|R)$ means "if $L$ occurs, do $M$; if $R$ occurs, do $M$".

(c) **(10 pts)** Calculate the expected payoffs for all combinations of pure strategies (it should be a $4 \times 4$ matrix).

The expected payoffs are given by these matrices:

| | $(M\|L, M\|R)$ | $(M\|L, B\|R)$ | $(B\|L, M\|R)$ | $(B\|L, B\|R)$ |
|---|---|---|---|---|
| $LL$ | $0.5HLM + 0.5LLM$ | $0.5HLM + 0.5LLM$ | $0.5HLB + 0.5LLB$ | $0.5HLB + 0.5LLB$ |
| $LR$ | $0.5HLM + 0.5LRM$ | $0.5HLM + 0.5LRB$ | $0.5HLB + 0.5LRM$ | $0.5HLB + 0.5LRB$ |
| $RL$ | $0.5HRM + 0.5LLM$ | $0.5HRB + 0.5LLM$ | $0.5HRM + 0.5LLB$ | $0.5HRB + 0.5LLB$ |
| $RR$ | $0.5HRM + 0.5LRM$ | $0.5HRB + 0.5LRB$ | $0.5HRM + 0.5LRM$ | $0.5HRB + 0.5LRB$ |

| | $(M\|L, M\|R)$ | $(M\|L, B\|R)$ | $(B\|L, M\|R)$ | $(B\|L, B\|R)$ |
|---|---|---|---|---|
| $LL$ | 0,0.5 | 0,0.5 | 2,0 | 2,0 |
| $LR$ | 1.5,0.5 | 2.5,-0.5 | 2.5, 1 | 3.5, 0 |
| $RL$ | 0.5,0.5 | 1.5,1 | 1.5,-0.5 | 2.5,0 |
| $RR$ | 2,0.5 | 4,0 | 2,0.5 | 4,0 |

(d) **(10 pts)** Find the set of pure strategy weak sequential equilibria.

The pure strategy NE are $(RR, (M|L, M|R))$ and $(LR, (B|L, M|R))$. We also need to find the beliefs that support these outcomes. For $(RR, (M|L, M|R))$, let the probability that Player 2 places on $H$ after observing Player 2 choosing $L$ be $p$. Then the expected payoff of playing $M$ must be greater than or equal to the expected payoff of playing $B$:

$$p(-1) + (1-p)2 \geq 0 \rightarrow p \leq \frac{2}{3}$$

For $(LR, (B|L, M|R))$, Player 2 places probability 1 on $H$ after observing Player 2 choosing $L$, and 1 on $L$ after observing Player 2 choosing $R$. The optimal actions given these beliefs are the same as in the NE.

The set of pure WSE are:

- $(RR, (M|L, M|R))$, where Player 2's beliefs at information set $\{HL, LL\}$ may be $(p, 1-p)$ with $0 \leq p \leq \frac{2}{3}$, and beliefs at information set $\{HR, LR\}$ are $(\frac{1}{2}, \frac{1}{2})$.
- $(LR, (B|L, M|R))$, where Player 2's beliefs at information set $\{HL, LL\}$ are $(1, 0)$ and beliefs at information set $\{HR, LR\}$ are $(0, 1)$.