CUR 412: Game Theory and its Applications

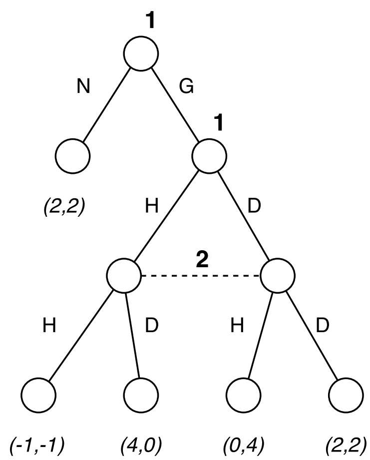
Final Exam

Ronaldo Carpio Jan. 13, 2015

Instructions:

- Please write your name in English.
- This exam is closed-book.
- Total time: 120 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(24 pts)** Consider the following extensive form game:



(a) **(8 pts)** Write down the matrix for the strategic form of this game.

	H	D
NH	2,2	2,2
ND	2,2	2,2
GH	-1,-1	4,0
GD	0,4	2,2

(b) **(8 pts)** Find all pure strategy Nash equilibria.

The pure NE are $(NH, H), (ND, H)$, and (GH, D).

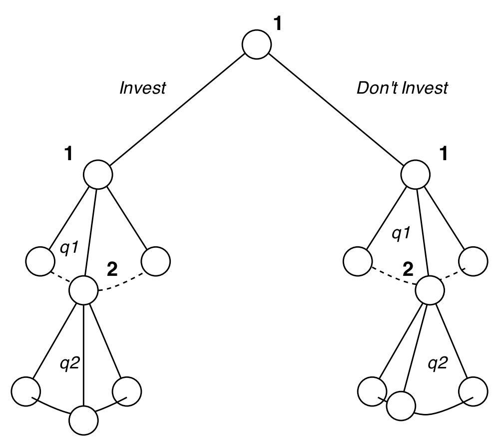
(c) **(8 pts)** Find all pure strategy subgame perfect NE.

The subgame after G is a simultaneous-move game equivalent to the Hawk-Dove game. The NE of the subgame are HD and DH, so (NH, H) cannot be subgame perfect. Therefore, the SPNE are (GH, D) and (ND, H).

Q2. **(24 pts)** Consider two firms that play a Cournot duopoly game with inverse demand $p = 100 - q$ and costs for each firm given by $c_i(q_i) = 10q_i$. Suppose that before the Cournot duopoly game, Firm 1 can choose to invest in *cost reduction*. If Firm 1 does, then it must pay a one-time cost of F, and its cost function drops to $c_1(q_1) = 5q_1$. If Firm 1 does not invest in cost reduction, there is no change.

(a) **(8 pts)** Write down the tree representation of this game.

The Cournot duopoly is a simultaneous-move game, where each player does not know what the other player has chosen when choosing his own action. One way to draw the tree diagram is as follows: Note that Player 2 has 2 information sets; he does not know what quantity Player 1 chose.



(b) **(12 pts)** Find the value of F for which the unique subgame perfect NE has Firm 1 investing. Call this F^*.

In the Cournot game, each firm chooses q_i to maximize profits $(100 - q_1 - q_2)q_i - c_i(q_i)$, taking q_j as given. Suppose Firm 1 does not invest in cost reduction. Then the profit-maximizing conditions are:

$$q_1 = \frac{90 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$$

which has a solution at $q_1 = q_2 = 30$, and profits are $\pi_1 = \pi_2 = 900$. If Firm 1 invests in cost reduction, then the profit-maximizing conditions are:

$$q_1 = \frac{95 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$$

which has a solution at $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and Firm 1's profits are 1111.11. Therefore, if $F < 1111.11 - 900 = 211.11$, it is optimal for Firm 1 to invest.

(c) **(4 pts)** Assume that $F > F^*$. Find a Nash equilibrium of this game that is not subgame perfect.

Any SPNE must satisfy the condition that the strategies induce a NE in each subgame. Therefore, in the subgame after *Invest*, any SPNE must result in $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and in the subgame after *Don't Invest*, any SPNE must result in $q_1 = q_2 = 30$. Any strategy profile that deviates from this in the *Invest* subgame will still be a NE if $F > F^*$, since that subgame will not actually be reached and will not affect the final payoff. Therefore, an example of a NE that is not SPNE is the following:

- Firm 1's strategy: (*Don't Invest*, $q_1 = \frac{100}{3}, q_1 = 30$), where the first q_1 specifies the action in the *Invest* subgame, and the second q_1 specifies the action in the *Don't Invest* subgame
- Firm 2's strategy: $(q_2 \neq \frac{85}{3}, q_2 = 30)$. Any choice of q_2 in the *Invest* subgame is possible; it does not affect the final payoff.

Q3. **(24 pts.)** Consider the following Bertrand duopoly model. The demand function for each firm $i = 1, 2$ as a function of prices P_1, P_2 is:

$$Q_i(P_1, P_2) = \begin{cases} 2 - P_i & \text{if } P_i < P_j \\ \frac{2-P_i}{2} & \text{if } P_i = P_j \\ 0 & \text{if } P_i > P_j \end{cases}$$

Suppose costs are zero, so each firm's profit is

$$\pi_i(P_1, P_2) = P_i Q_i(P_1, P_2)$$

Let $P^m = 1$ be the price that would be chosen in monopoly (i.e. it maximizes $P(2 - P)$). Suppose that this game is repeated infinitely, in period $t = 1, 2, ...$ The payoff of firm i to the infinite sequence of profits $\{\pi_{i,t}\}$ is the discounted average (where $0 \leq \delta \leq 1$):

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_{i,t}$$

Consider this strategy profile:

- Choose P^m in the first period, and after any history in which both firms have always played P^m.
- Choose $P_i = 0$ after any other history.

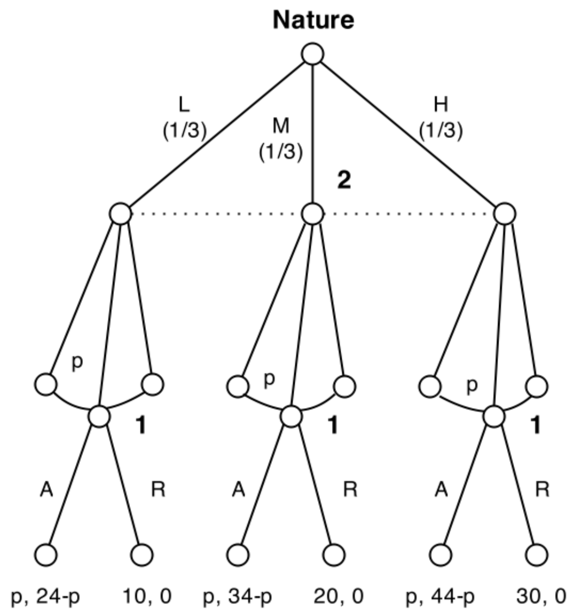
(a) **(8 pts)** Calculate the 2x2 matrix of payoffs for the single stage game, where each firm chooses either P^m or 0.

	p^m	*0*
p^m	1/2,1/2	0,0
0	0,0	0,0

(b) **(16 pts)** For what range of δ, if any, is the above strategy profile a SPNE in the infinitely repeated game?

Suppose both players do not deviate from the strategy. The sequence of payoffs is $\frac{1}{2}, \frac{1}{2}, ...$ with a discounted average of $\frac{1}{2}$. If one player deviates by playing 0, then his payoff sequence will be 0, with a discounted average of 0. Therefore, it is not optimal to deviate for any $0 < \delta < 1$.

Q4. **(28 pts.)** Consider this extensive-form game: Suppose Player 1 owns a car. Nature



chooses the quality of the car, which may be high (H), medium (M), or low (L), with equal probabilities of $\frac{1}{3}$ each. Player 1 knows what the result of Nature's choice is, but Player 2 does not. Player 2 is deciding how much to offer for the car. The value of the car to each player is:

$$v_1(q) = \begin{cases} 20 & \text{if } q = L \\ 30 & \text{if } q = M \\ 40 & \text{if } q = H \end{cases}$$

$$v_2(q) = \begin{cases} 24 & \text{if } q = L \\ 34 & \text{if } q = M \\ 44 & \text{if } q = H \end{cases}$$

The sequence of actions is:

1. Player 2 offers a price p to Player 1.
2. Player 1 chooses whether to *Accept* or *Reject*.
    - If Player 1 *Accepts*, Player 1's payoff is the price p, and Player 2's payoff is his valuation of the car minus the price p.
    - If Player 1 *Rejects*, Player 1's payoff is his valuation of the car, and Player 2's payoff is 0.

(a) **(8 pts)** Suppose Player 2 offers $p \geq 0$. Find the best response of Player 1, for each of the three possible quality levels.

- If $q = L$, then Player 1 should *Accept* if $p \geq 20$.

- If $q = M$, then Player 1 should *Accept* if $p \geq 30$.
- If $q = H$, then Player 1 should *Accept* if $p \geq 40$.

(b) **(10 pts)** Show that there is no pure strategy weak sequential equilibrium in which the car is traded at a price equal to 30, the expected value of v_1.

We can find the solution using backwards induction. Part (a) gives the best response of Player 1 in the last stage. Player 2's beliefs over $\{L, M, H\}$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ matching the distribution of Nature's choice. Taking that as given, we can find the expected payoff for Player 2 for each price $p, E_2(p)$:

- If $p < 20$, Player 1 will reject in all cases. $E_2(p) = 0$.
- If $20 \leq p \leq 24, E_2(p) = \frac{1}{3}(24 - p) + \frac{1}{3}(0) + \frac{1}{3}(0) = \frac{24-p}{3}$. This is maximized if $p = 20$.
- If $p > 24$, the expected payoff is negative.

Player 2's expected payoff from offering a price of $p = 30$ is $\frac{1}{3}(24 - 30) + \frac{1}{3}(34 - 30) = -\frac{2}{3}$, which is worse than getting a payoff of 0 (which can always be achieved by offering $p < 20$). Therefore, Player 2 will not offer $p = 30$.

(c) **(10 pts)** Find the range of p at which a trade can occur in a pure strategy weak sequential equilibrium.

As in part (b), Player 2's best response is $p = 20$. This is the only price that is part of a weak sequential equilibrium.