CUR 412: Game Theory and its Applications

Final Exam

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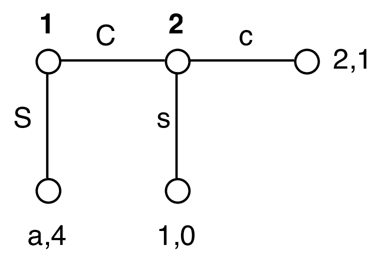
Instructions:

- Please write your name in English.
- This exam is closed-book.
- Total time: 120 minutes.
- There are 4 questions, for a total of 100 points.

Q1. **(20 pts)** Consider the following extensive form game:



(a) **(10 pts)** Assume $a = 1$. Find the set of pure strategy NE and subgame perfect NE.

There is one SPNE: (C, c). The strategic form of this game is given by:

	s	c
S	1,4	1,4
C	1,0	2,1

The set of NE is (S, s) and (C, c).

(b) **(5 pts)** Find the range of a for which S is the unique subgame perfect equilibrium outcome.

Player 1's only optimal choice is S if $a > 2$.

(c) **(5 pts)** Find the range of a for which (C, c) is the unique Nash equilibrium outcome.

The strategic form game is given by:

	s	c
S	a,4	a,4
C	1,0	2,1

(C, c) is the unique NE if $a < 2$ and $a < 1$, or simply if $a < 1$.

Q2. **(20 pts)** Consider the following Cournot duopoly game. Firms 1 and 2 choose output levels q_1, q_2; the profit function of firm i is:

$$\pi_i(q_1, q_2) = \begin{cases} q_i(1 - q_1 - q_2) & \text{if } q_1 + q_2 \leq 1 \\ 0 & \text{if } q_1 + q_2 > 1 \end{cases}$$

Firm 2 is run by its owner, while Firm 1 is run by a *manager* whose utility function is given by:

$$w(q_1, q_2) = \pi_1(q_1, q_2) + \alpha q_1$$

where $0 \leq \alpha \leq 1$. The sequence of actions is as follows:

1. First, the owner of firm 1 chooses $\alpha \in [0, 1]$, which is known by all players.
2. Second, the manager of firm 1 and the owner of firm 2 simultaneously choose q_1, q_2, respectively.

The owners of each firm want to maximize their profits, π_i. The manager wants to maximize his payoff w. Find the subgame perfect equilibrium levels of α, q_1, q_2.

In the last stage, take α as given. We will find the NE of the resulting Cournot duopoly. The manager of Firm 1 chooses q_1 to maximize

$$q_1(1 - q_1 - q_2) + \alpha q_1$$

which is maximized at $q_1 = \frac{1+\alpha-q_2}{2}$. The owner of Firm 2 chooses q_2 to maximize

$$q_2(1 - q_1 - q_2)$$

which is maximized at $q_2 = \frac{1-q_1}{2}$. The NE is given by the solution to these two equations, which is

$$q_1 = \frac{1 + 2\alpha}{3}, q_2 = \frac{1 - \alpha}{3}$$

In the first stage, the owner of Firm 1 will take these as given, and choose α to maximize

$$q_1(1 - q_1 - q_2) = \frac{1 + 2\alpha}{3}\left(1 - \frac{1 + 2\alpha}{3} - \frac{1 - \alpha}{3}\right)$$

$$= \frac{(1 - \alpha)(1 + 2\alpha)}{9}$$

which is maximized when $\alpha = 1/4$. Therefore, the SPNE is:

$$\alpha = \frac{1}{4}, q_1 = \frac{1}{2}, q_2 = \frac{1}{4}$$

Q3. **(30 pts.)** Suppose two firms in a Cournot duopoly have zero unit cost and fixed cost. Each firm chooses q_1, q_2, respectively. Market demand is given by $P = 200 - Q$, where $Q = q_1 + q_2$.

(a) **(10 pts.)** Find the Nash equilibrium levels of q_1, q_2, and firms' profits.

Each firm chooses q_i to maximize $q_i(200 - q_i - q_j)$, which is maximized at $q_i = \frac{200-q_j}{2}$. The NE is when $q_1 = q_2 = \frac{200}{3}$. $P = \frac{200}{3}$, each firm's profit is $\frac{40000}{9} \sim 4444$.

(b) **(5 pts.)** Suppose both firms combined into a single monopolist. Find the equilibrium price and quantity.

The monopolist chooses q to maximize $q(200-q)$, which is maximized at $q = 100$. $P = 100$, profit is 10000.

Now, suppose this game is infinitely repeated, with discount factor $\delta < 1$. In each period, a firm can choose to:

- *Collude*, in which case the firm chooses to produce half of the monopolist's quantity in (b), or
- *Defect*, in which case the firm maximizes its own profits, given the other firm's quantity.

(c) **(5 pts.)** Write down the 2×2 matrix of payoffs for a single stage of the repeated game.

Suppose Player 1 chooses *Defect* while Player 2 chooses *Collude*. Player 2 will choose $q_2 = 50$. Player 1's best response is to choose q_1 to maximize $q_1(200 - q_1 - 50)$, which is maximized when $q_1 = 75$. Profits for Player 1 will be 5625, for Player 2 it will be 3750. The 2×2 matrix is:

	Collude	*Defect*
Collude	5000, 5000	3750, 5625
Defect	5625, 3750	4444, 4444

This is a Prisoner's Dilemma.

(d) **(10 pts.)** Find the range of δ for which it is a subgame perfect Nash equilibrium when both firms play a modified grim trigger strategy:

- If *Defect* has never been played by either firm, then choose *Collude*.
- If *Defect* has been played at any time in the past by either firm, then choose *Defect*.

We classify all histories into two cases:

- Case 1: *Defect* has never been played by either player, so Modified Grim Trigger will play *Collude*. If there is no deviation, the sequence of outcomes will be (*Collude*, *Collude*) forever, which gives a discounted average of 5000 to both players.

  Suppose Player 1 plays a one-shot deviation by playing *Defect*, then reverting to Modified Grim Trigger afterwards. The sequence of outcomes will be (*Defect*, *Collude*), followed by $(Defect, Defect)$ forever. Player 1's sequence of payoffs is $5625, 4444, 4444, ...$ which gives a discounted average of

$$= (1-\delta)(5625 + \delta 4444 + \delta^2 4444 + ...)$$

$$= (1-\delta)(5625 + 4444\frac{\delta}{1-\delta})$$

$$= (1-\delta)5625 + 4444\delta$$

This deviation is profitable if

$$(1-\delta)5625 + 4444\delta \geq 5000$$

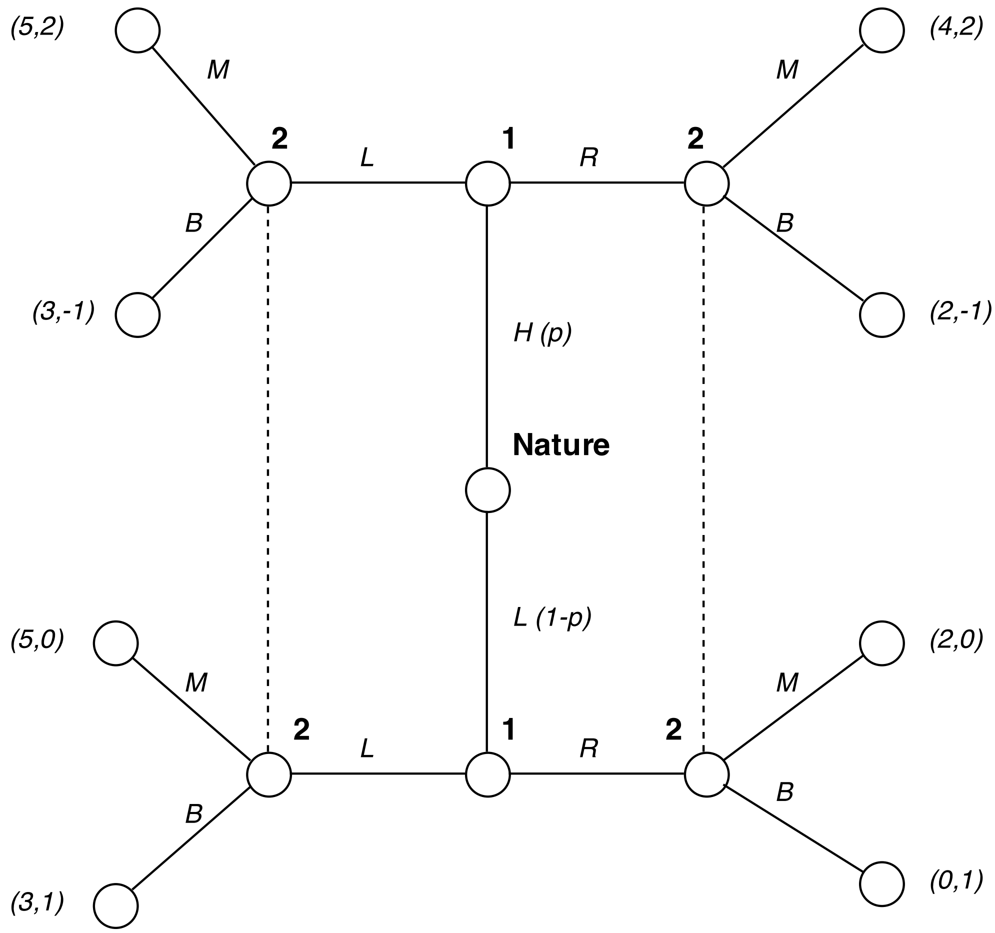
or if $\delta \leq 0.529$.

- Case 2: $Defect$ has been played by some player, so Modified Grim Trigger will play $Defect$. If there is no deviation, the sequence of outcomes will be $(Defect, Defect)$ forever, which gives a discounted average of 4444 to both players.

  Suppose Player 1 plays a one-shot deviation by playing $Collude$, then reverting to Modified Grim Trigger afterwards. The sequence of outcomes will be $(Collude, Defect)$ followed by $(Defect, Defect)$ forever. Player 1's sequence of payoffs is $3750, 4444, 4444, ...$ This gives a lower discounted average than not deviating for any value of δ, so there is no profitable deviation for this case.

Therefore, a profitable one-shot deviation does not exist if $\delta \geq 0.529$, and therefore the given strategy profile is a SPNE.

Q4. **(30 pts.)** Consider this signaling game. Nature chooses H, L with probability $p = \frac{1}{2}$. Player 1's payoff is listed first in the pair of numbers for each outcome.



(a) **(3 pts)** For Player 1 and Player 2, list the histories in each player's information sets.

- Player 1: $\{H\}, \{L\}$
- Player 2: $\{HL, LL\}, \{HR, LR\}$

(b) **(3 pts)** For each of Player 1 and Player 2's information sets, list their pure strategies.

- Player 1: For information set H, pure strategies are L, R. For information set L, pure strategies are L, R. Overall, Player 1 has 4 pure strategies: LL, LR, RL, RR.
- Player 2: For information set $\{HL, LL\}$, pure strategies are M, B. For information set $\{HR, LR\}$, pure strategies are M, B. Overall, Player 2 has 4 pure strategies: $(M|R, M|L)$, $(M|R, B|L)$, $(B|R, M|L)$, $(B|R, B|L)$.

(c) **(12 pts)** Calculate the expected payoffs for all combinations of pure strategies (it should be a 4×4 matrix).

(d) **(12 pts)** Find the set of pure strategy weak sequential equilibria.

	$M\|R, M\|L$	$M\|R, B\|L$	$B\|R, M\|L$	$B\|R, B\|L$
RR	3,1	3,1	1,0	1,0
RL	4.5, 1	3.5, 1.5	3.5, -0.5	2.5, 0
LR	3.5, 1	2.5, -0.5	2.5, 1.5	1.5, 0
LL	5, 1	3, 0	5, 1	3, 0

The NE are: $(RL, (M|R, B|L))$, $(LL, (M|R, M|L))$, and $(LL, (B|R, M|L))$. Going through each one in turn:

- $(RL, (M|R, B|L))$: Player 2's beliefs at information set after Player 1 chooses R must be $(1, 0)$ (probability 1 on H). After Player 1 chooses L, beliefs must be $(0, 1)$. This is a WSE.
- $(LL, (M|R, M|L))$: Player 2's beliefs after Player 1 chooses L must be $(p, 1-p) = (0.5, 0.5)$. The information set after Player 1 chooses R is not reached with positive probability, so any beliefs are consistent. Denote beliefs as $(q, 1-q)$, where q is the probability on H. The range of q that makes M optimal is:

  $$q2 + (1-q)0 \geq q(-1) + (1-q)1$$

  or if $q \geq \frac{1}{4}$.
- $(LL, (B|R, M|L))$: Player 2's beliefs after Player 1 chooses L must be $(p, 1-p) = (0.5, 0.5)$. The information set after Player 1 chooses R is not reached with positive probability, so any beliefs are consistent. Denote beliefs as $(q, 1-q)$, where q is the probability on H. The range of q that makes B optimal is:

  $$q2 + (1-q)0 \leq q(-1) + (1-q)1$$

  or if $q \leq \frac{1}{4}$.