1 Solutions to Homework 1

1.1 16.1 (Working on a joint project)

Suppose the two actions are W for *Work hard*, G for *Goof off*. Any matrix where $u_i(W,W) > u_i(G,W) > u_i(G,G) > u_i(W,G)$ for both players $i = 1,2$ will satisfy the conditions. For example:

	W	G
W	4, 4	0, 3
G	3, 0	1, 1

1.2 27.1 (Variant of Prisoner's Dilemma with altruistic preferences)

The payoff matrix is as follows:

	Q	F
Q	$2+2\alpha, 2+2\alpha$	$0+3\alpha, 3$
F	$3, 0+3\alpha$	$1+\alpha, 1+\alpha$

If $\alpha = 1$, this becomes

	Q	F
Q	4,4	3,3
F	3,3	2,2

which is not the Prisoner's Dilemma, since $u_1(Q,F) = u_1(F,Q)$.

For this game to be the same as the Prisoner's Dilemma, for both players $i = 1,2$, we should have $u_i(F,Q) > u_i(Q,Q) > u_i(F,F) > u_i(Q,F)$. This gives us

$$3 > 2 + 2\alpha > 1 + 1\alpha > 3\alpha$$

which is satisfied (for non-negative α) if $\alpha < 1/2$.

1.3 31.2 (Hawk-Dove)

Let the actions be P for *Passive*, A for *Aggressive*.

- Each player prefers A if its opponent is $P \rightarrow u_i(A,P) > u_i(P,P)$
- Each player prefers P if its opponent is $A \rightarrow u_i(P,A) > u_i(A,A)$
- Given its own stance, each player prefers its opponent to be $P \rightarrow u_i(A,P) > u_i(A,A)$ and $u_i(P,P) > u_i(P,A)$.

Combined, the preferences over outcomes for each player $i = 1,2$ are $u_i(A,P) > u_i(P,P) > u_i(P,A) > u_i(A,A)$. An example of a payoff matrix that satisfies these preferences is:

	P	A
P	2,2	1,3
A	3,1	0,0

1.4 47.1 (Strict equilibria and dominated actions)

For player 1, T is weakly dominated by M, and strictly dominated by B. For player 2, no action is weakly or strictly dominated. The game has a unique Nash equilibrium, (M, L). This equilibrium is not strict. (When player 2 chooses L, B yields player 1 the same payoff as M.)

1.5 48.1 (Voting)

First, consider an outcome where the winner has exactly one more vote than the loser, and at least one citizen voted for the winner, but actually prefers an outcome where the loser wins. Taking the other players' actions as given, this citizen can change the election result by switching his vote, therefore this is not a Nash equilibrium.

Second, consider an outcome where the winner has exactly one more vote than the loser, but no citizens in the previous case exist (i.e. all citizens who voted for the winner, prefer that outcome). If one of those citizens switched his vote to the loser, the election result would change, but none of those citizens have an incentive to do so. Any citizen who prefers the loser to win is already voting for the loser, so cannot change the election result. Therefore, this is a Nash equilibrium.

Third, consider an outcome where the winner has three or more votes than the loser (there cannot be a margin of two votes, since there are an odd number of citizens). No citizen can change the election result through his action alone, so no one has an incentive to deviate. Therefore, this is also a Nash equilibrium.

As explained in the textbook, voting for the less preferred candidate is a weakly dominated action. If everybody voted for their preferred candidate, candidate A would win (by assumption on the number of voters) and the outcome is a Nash equilibrium where no one plays a weakly dominated action.

1.6 58.1 Cournot's duopoly game with linear inverse demand and different unit costs

Following the analysis in the text, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} (\alpha - c_1 - q_2)/2 & \text{if } q_2 \leq \alpha - c_1 \\ 0 & \text{otherwise} \end{cases}$$

while that of firm 2 is

$$b_2(q_1) = \begin{cases} (\alpha - c_2 - q_1)/2 & \text{if } q_1 \leq \alpha - c_2 \\ 0 & \text{otherwise} \end{cases}$$

To find the Nash equilibrium, first plot these two functions. Each function has the same general form as the best response function of either firm in the case studied in the text. However, the fact that $c_1 \neq c_2$ leads to two qualitatively different cases when we combine the two functions to find a Nash equilibrium. If c_1 and c_2 do not differ very much then the functions in the analogue of Figure 59.1 in the textbook (the plot of best response functions) intersect at a pair of outputs that are both positive. If c_1 and c_2 differ a lot, however, the functions intersect at a pair of outputs in which $q_1 = 0$.

Precisely, if $c_1 \leq (\alpha + c_2)/2$ then the downward-sloping parts of the best response functions intersect (as in Figure 59.1 in the textbook), and the game has a unique Nash equilibrium, given by the solution of the two equations

$$q_1 = (\alpha - c_1 - q_2)/2$$

$$q_2 = (\alpha - c_2 - q_1)/2$$

This solution is:

$$(q_1^*, q^*)2) = ((\alpha - 2c_1 + c_2)/3, (\alpha - 2c_2 + c_1)/3)$$

If $c_1 > (\alpha + c_2)/2$ then the downward-sloping part of firm 1's best response function lies below the downward-sloping part of firm 2's best response function (as in Figure 12.1), and the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, (\alpha - c_2)/2)$.

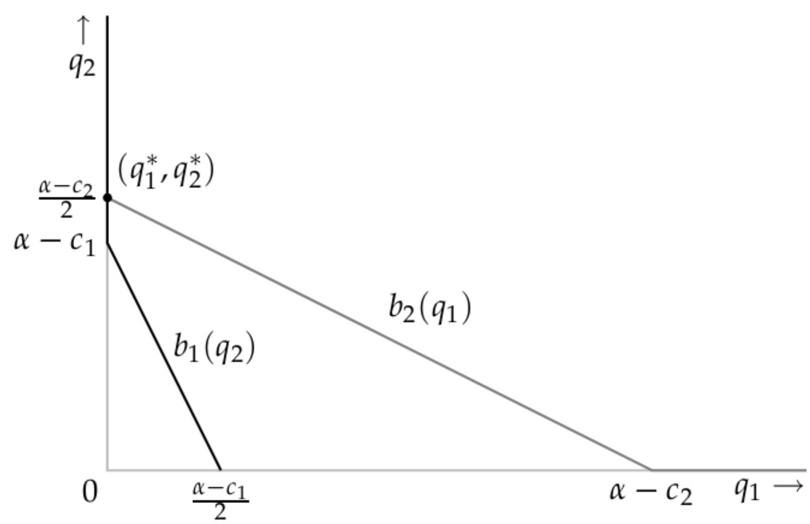

Figure 12.1 The best response functions in Cournot's duopoly game under the assumptions of Exercise 58.1 when $\alpha - c_1 < \frac{1}{2}(\alpha - c_2)$. The unique Nash equilibrium in this case is $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

In summary, the game always has a unique Nash equilibrium, defined as follows:

$$\begin{cases} ((\alpha - 2c_1 + c_2)/3, (\alpha - 2c_2 + c_1)/3) & \text{if } c_1 \leq (\alpha + c_2)/2 \\ (0, (\alpha - c_2)/2) & \text{otherwise} \end{cases}$$

The output of firm 2 exceeds that of firm 1 in every equilibrium.

If c_2 decreases then firm 2's output increases and firm 1's output either falls, is $c_1 \leq (\alpha+c_2)/2$, or remains equal to 0, if $c_1 > (\alpha + c_2)/2$. The total output increases and the price falls.

1.7 61.1 (Cournot's game with many firms)

Suppose there are n firms with identical unit cost c. As before, the best response of firm 1 will be

$$b_1(q_2, ...q_n) = \begin{cases} q_1 \cdot (\alpha - c - q_2 - ... - q_n) & \text{if } q_1 + ... + q_n \leq \alpha \\ -cq_1 & \text{if } q_1 + ... + q_n > \alpha \end{cases}$$

There are n equations in the conditions for Nash equilibrium:

$$q_1^* = b_1(q_2^*, ...q_n^*)$$

$$q_2^* = b_2(q_1^*, ...q_n^*)$$

$$...$$

$$q_n^* = b_n(q_1^*, ...q_{n-1}^*)$$

which becomes (setting $Q = q_1 + q_2 + ... + q_n$)

$$q_1^* = \frac{1}{2}(\alpha - c - Q + q_1)$$

$$q_2^* = \frac{1}{2}(\alpha - c - Q + q_2)$$

$$...$$

$$q_n^* = \frac{1}{2}(\alpha - c - Q + q_n)$$

We could use linear algebra to solve this system of equations, but a simple way is to rearrange each equation so that

$$q_i^* = \alpha - c - Q$$

This shows that all firms choose the same level of output in equilibrium. Total output $Q = nq_i = \frac{n}{n+1}(\alpha - c)$ and $q_i = \frac{\alpha - c}{n+1}$. The market price is:

$$P(Q) = \alpha - Q = \frac{\alpha}{n+1} + \frac{n}{n+1}c$$

Taking the limit as $n \to \infty$, the first term goes to 0 and the second term goes to c. Therefore, as the number of firms becomes very large, the Nash equilibrium becomes the same as the perfectly competitive equilibrium outcome.

1.8 67.2 (Bertrand's duopoly game with discrete prices)

In this game, actions are restricted to the integers $0, ...c-1, c, c+1, c+2, ...$

Let's consider the possible cases:

- $p_1 = p_2 = c$. Both firms are making zero profit. If firm i lowered its price to below c, it would make a negative profit. If it raised its price to above c, it would not get any customers and hence make zero profit. As in the original game, there is no incentive to deviate, therefore (c, c) is a Nash equilibrium.
- $p_i < c$ for either firm. The firm with the lowest price is making negative profit, and can improve to zero profit by setting $p_i = c$. Not a Nash equilibrium.
- $p_i = p_2 = c+1$. Both firms are making positive profit. If firm i lowered its price, it would make zero or negative profit. If it raises its price, it makes zero profit. Therefore, $(c+1, c+1)$ is also a Nash equilibrium.
- $p_i = c, p_j > c$. Firm i is making zero profit, can improve by raising its price to $c+1$. Not a Nash equilibrium.
- $p_i > p_j \geq c+1$. Firm i is making zero profit, can improve by lowering price to equal p_j. Not a Nash equilibrium.

As compared to the original formulation of Bertrand duopoly, there is an additional Nash equilibrium because we have eliminated some possible actions that would give an incentive to deviate.