1 Solutions to Homework 2

1.1 76.1 (Competition in product characteristics)

Suppose there are two firms. If the products are different, then either firm increases its market share by making its product more similar to that of its rival. Thus in every possible equilibrium the products are the same. But if $x_1 = x_2 \neq m$ then each firm's market share is 50%, while if it changes its product to be closer to m then its market share rises above 50%. Thus the only possible equilibrium is $(x_1, x_2) = (m, m)$. This pair of positions is an equilibrium, since each firm's market share is 50%, and if either firm changes its product its market share falls below 50%.

Now suppose there are three firms. If all firms' products are the same, each obtains one-third of the market. If $x_1 = x_2 = x_3 = m$ then any firm, by changing its product a little, can obtain close to one-half of the market. If $x_1 = x_2 = x_3 \neq m$ then any firm, by changing its product a little, can obtain more than one-half of the market. If the firms' products are not all the same, then at least one of the extreme products is different from the other two products, and the firm that produces it can increase its market share by making it more similar to the other products. Thus when there are three firms there is no Nash equilibrium.

1.2 80.1 (Timing product release)

Let the actions of each player be the time chosen: t_i, t_j.

Case 1: $h(t_j) < 1/2$. If player i chooses:

- $t_i = t_j$, both players get a share of $1/2$.
- $t_i < t_j$, player i gets a share of $h(t_i) < 1/2$.
- $t_i > t_j$, player i gets a share of $1 - h(t_j) > 1/2$.

In this case, the best response is any $t_i > t_j$.

Case 2: $h(t_j) = 1/2$. If player i chooses:

- $t_i = t_j$, both players get a share of $1/2$.
- $t_i < t_j$, player i gets a share of $h(t_i) < 1/2$.
- $t_i > t_j$, player i gets a share of $1 - h(t_j) = 1/2$.

Best response is $t_j \geq t_j$.

Case 3: $h(t_j) > 1/2$. If player i chooses:

- $t_i = t_j$, both players get a share of $1/2$.
- t_i slightly below t_j, player i gets a share of $h(t_i) > 1/2$.

- $t_i > t_j$, player i gets a share of $1 - h(t_j) < 1/2$.

A best response does not exist in this case, because it's always possible to increase t_i by a small amount without hitting t_j. The only Nash equilibrium is $(\frac{1}{2}, \frac{1}{2})$.

1.3 85.1 (Second-price sealed-bid auction with two bidders)

If player 2's bid b_2 is less than v_1 then any bid of b_2 or more is a best response of player 1 (she wins and pays the price b_2). If player 2's bid is equal to v_1 then every bid of player 1 yields her the payoff zero (either she wins and pays v_1, or she loses), so every bid is a best response. If player 2's bid b_2 exceeds v_1 then any bid of less than b_2 is a best response of player 1. (If she bids b_2 or more she wins, but pays the price $b_2 > v_1$, and hence obtains a negative payoff.) In summary, player 1's best response function is

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & \text{if } b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & \text{if } b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & \text{if } b_2 > v_1. \end{cases}$$

By similar arguments, player 2's best response function is

$$B_2(b_1) = \begin{cases} \{b_2 : b_2 > b_1\} & \text{if } b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & \text{if } b_1 = v_2. \\ \{b_2 : 0 \leq b_2 \leq b_1\} & \text{if } b_1 > v_2. \end{cases}$$

These best response functions are shown in Figure 16.1.

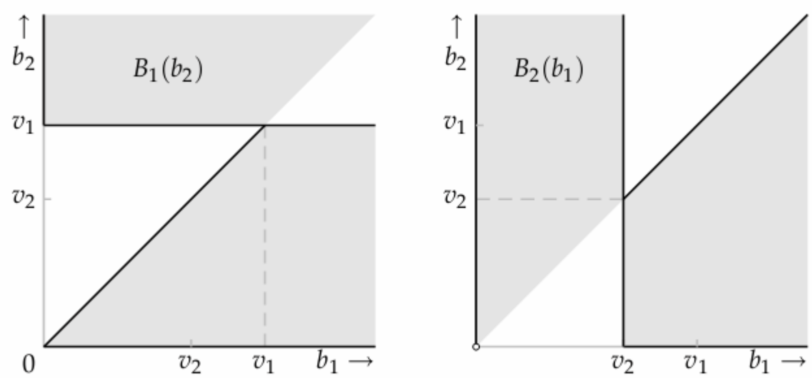

Figure 16.1 The players' best response functions in a two-player second-price sealed-bid auction (Exercise 86.1). Player 1's best response function is in the left panel; player 2's is in the right panel. (Only the edges marked by a black line are included.)

Superimposing the best response functions, we see that the set of Nash equilibria is the shaded set in Figure 17.1, namely the set of pairs (b_1, b_2) such that either

$$b_1 \leq v_2 \text{ and } b_2 \geq v_1$$

or

$$b_1 \geq v_2,\ b_1 \geq b_2, \text{ and } b_2 \leq v_1.$$

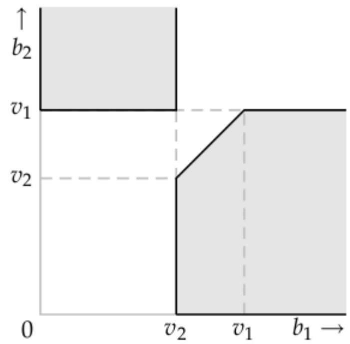

Figure 17.1 The set of Nash equilibria of a two-player second-price sealed-bid auction (Exercise 86.1).

1.4 101.1 (Variant of Matching Pennies)

The analysis is the same as for Matching Pennies. There is a unique steady state, in which each player chooses each action with probability $\frac{1}{2}$.

1.5 110.1 (Expected Payoffs)

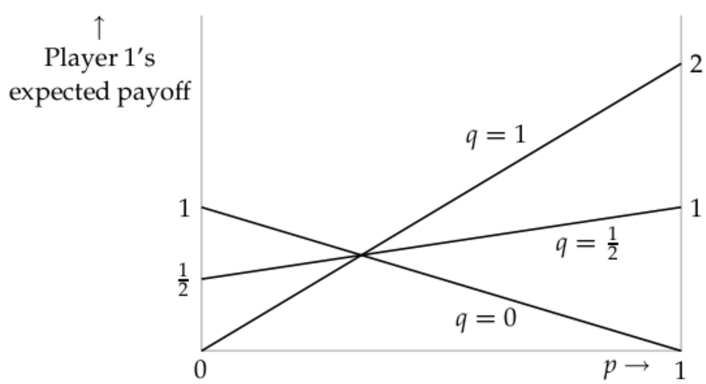

Figure 24.1 Player 1's expected payoff as a function of the probability p that she assigns to B in *BoS*, when the probability q that player 2 assigns to B is 0, $\frac{1}{2}$, and 1.

For *BoS*, player 1's expected payoff is shown in Figure 24.1.

For the game in the right panel of Figure 21.1 in the book, player 1's expected payoff is shown in Figure 24.2.

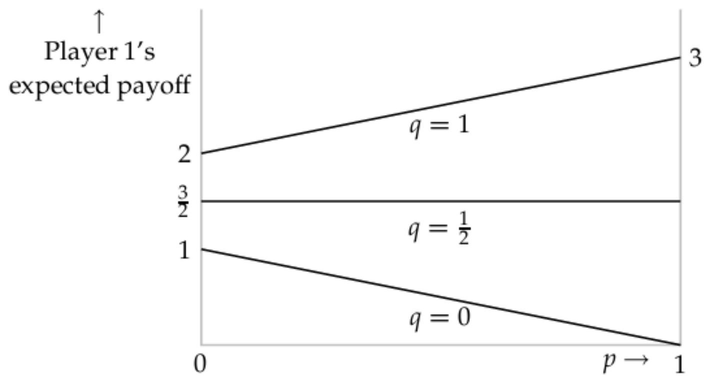

Figure 24.2 Player 1's expected payoff as a function of the probability p that she assigns to *Refrain* in the game in the right panel of Figure 21.1 in the book, when the probability q that player 2 assigns to *Refrain* is 0, $\frac{1}{2}$, and 1.

1.6 111.1 (Best Responses)

For the first game:

- If $q = 0$, the best response is $p = 0$.
- If $q = \frac{1}{2}$, the best response is $p = 1$.
- If $q = 1$, the best response is $p = 1$.

For the second game:

- If $q = 0$, the best response is $p = 0$.
- If $q = \frac{1}{2}$, any $p, 0 \leq p \leq 1$ is a best response.
- If $q = 1$, the best response is $p = 1$.

1.7 114.2 (Games with mixed strategy equilibria)

Let p be the probability that Player 1 plays T, q be the probability that Player 2 plays L.

	L	R
T	6,0	0,6
B	3,2	6,0

There are no NE in pure strategies. For player 1, taking q as given:

- $E_1(T, q) = q \cdot 6 + (1 - q) \cdot 0 = 6q$

- $E_1(B,q) = q \cdot 3 + (1-q) \cdot 6 = 6 - 3q$

If $6q > 6 - 3q \Leftrightarrow q > \frac{2}{3}$, then $E_1(T,q) > E_1(B,q)$ and playing T with 100% probability is optimal. The best response function for p is:

$$\begin{cases} \{0\} & \text{if } q < 2/3 \\ \{p : 0 \leq p \leq 1\} & \text{if } q = 2/3 \\ \{1\} & \text{if } q > 2/3 \end{cases}$$

For player 2, taking p as given:

- $E_2(L,p) = p \cdot 0 + (1-p) \cdot 2 = 2 - 2p$
- $E_2(R,p) = p \cdot 6 + (1-p) \cdot 0 = 6p$

If $2 - 2p > 6p \Leftrightarrow p < \frac{1}{4}$, then $E_2(L,p) > E_2(R,p)$ and playing L with 100% probability is optimal. The best response function for q is:

$$\begin{cases} \{1\} & \text{if } p < 1/4 \\ \{q : 0 \leq q \leq 1\} & \text{if } p = 1/4 \\ \{0\} & \text{if } p > 1/4 \end{cases}$$

$p = \frac{1}{4}, q = \frac{2}{3}$ is the only mixed strategy NE.

	L	R
T	0,1	0,2
B	2,2	0,1

(B,L) and (T,R) are NE in pure strategies. For player 1, taking q as given:

- $E_1(T,q) = q \cdot 0 + (1-q) \cdot 0 = 0$
- $E_1(B,q) = q \cdot 2 + (1-q) \cdot 0 = 2q$

If $q > 0$, then $E_1(B,q) > E_1(T,q)$ and playing $Defender$ with 100% probability is optimal. The best response function for p is:

$$\begin{cases} \{p : 0 \leq p \leq 1\} & \text{if } q = 0 \\ \{0\} & \text{if } q > 0 \end{cases}$$

For player 2, taking p as given:

- $E_2(L,p) = p \cdot 1 + (1-p) \cdot 2 = 2 - p$
- $E_2(R,p) = p \cdot 2 + (1-p) \cdot 1 = 1 + p$

If $2 - p > 1 + p \Leftrightarrow p < \frac{1}{2}$, then $E_2(L,p) > E_2(R,p)$ and playing L with 100% probability is optimal. The best response function for q is:

$$\begin{cases} \{1\} & \text{if } p < 1/2 \\ \{q : 0 \leq q \leq 1\} & \text{if } p = 1/2 \\ \{0\} & \text{if } p > 1/2 \end{cases}$$

In addition to the two NE in pure strategies, there is a continuum of mixed strategy NE if $p \geq \frac{1}{2}, q = 0$.

1.8 118.3 (Defending Territory)

There are two players, *Attacker* and *Defender*. We will assign a payoff of 1 to the winner and 0 to the loser. The payoff matrix of the game without randomization is:

		Attacker		
		(0, 2)	(1, 1)	(2, 0)
Defender	(0, 3)	1,0	0,1	0,1
	(1, 2)	1,0	1,0	0,1
	(2, 1)	0,1	1,0	1,0
	(3, 0)	0,1	0,1	1,0

Like Matching Pennies, this is a constant-sum game, so there is no Nash equilibrium in pure strategies. Note that for $Defender$, $(1,2)$ weakly dominates $(0,3)$ and $(2,1)$ weakly dominates $(3,0)$.

First, we show that *Attacker* will not play $(1,1)$ in any MSNE. Let q_1, q_2, q_3 denote *Attacker*'s probability of playing $(0,2)$, $(1,1)$, $(2,0)$ respectively, with $q_1 + q_2 + q_3 = 1$. *Attacker* must play $(0,2)$ and $(2,0)$ with positive probability; otherwise, $Defender$ has an action that guarantees victory. Suppose *Attacker* plays $(1,1)$ with positive probability ($q_2 > 0$). $Defender$'s expected payoff to his pure strategies are:

- $(0,3) : q1$
- $(1,2) : q1 + q2$
- $(2,1) : q2 + q3$
- $(3,0) : q3$

$Defender$'s best response is to place zero probability on $(0,3)$ and $(3,0)$, since $(1,2)$ and $(2,1)$ give a strictly higher expected payoff. But if $Defender$ does not play $(0,3)$ and $(3,0)$, then $(1,1)$ becomes strictly dominated for *Attacker*, and therefore cannot be played in equilibrium. This is a contradiction, so the assumption must be false: *Attacker* does not play $(1,1)$ with positive probability.

If we assume *Attacker* will not play $(1,1)$, then $Defender$'s actions:

- $(0,3)$ and $(1,2)$ have the same payoffs
- $(2,1)$ and $(3,0)$ have the same payoffs

Attacker will choose q_1, q_3 to make $Defender$ indifferent between his actions that are played with positive probability. $Defender$'s expected payoff to playing $(0,3)$ or $(1,2)$: q_1 $Defender$'s expected payoff to playing $(2,1)$ or $(0,3)$: q_3 Equalizing them gives $q_1 = q_3 = 0.5$.

Now, consider $Defender$'s strategy. Assume the probability placed on $(0,3)$, $(1,2)$, $(2,1)$, $(0,3)$ are p_1, p_2, p_3, p_4 respectively, where $p_1 + p_2 + p_3 + p_4 = 1$. We can eliminate *Attacker*'s action $(1,1)$. For $Defender$, $(0,3)$ is equivalent in payoff to $(1,2)$, so for a given level of $p_1 + p_2$, the values of p_1 and p_2 are irrelevant. Likewise, $(2,1)$ is equivalent in payoff to $(0,3)$, so for a given level of $p_3 + p_4$, the values of p_3 and p_4 are irrelevant. The conditions for MSNE must be satisfied: $E_{Attacker}(1,1) \leq E_{Attacker}(0,2) = E_{Attacker}(2,0)$

- $E_{Attacker}(0,2) = p_3 + p_4$
- $E_{Attacker}(1,1) = p_1 + p_4$
- $E_{Attacker}(2,0) = p_1 + p_2$

$$E_{Attacker}(0,2) = E_{Attacker}(2,0) \Rightarrow p_1 + p_2 = p_3 + p_4$$

Since $p_1 + p_2 + p_3 + p_4 = 1$, then $p_1 + p_2 = p_3 + p_4 = 0.5$.

$$E_{Attacker}(1,1) \leq E_{Attacker}(0,2) \Rightarrow p_1 + p_4 \leq p_3 + p_4 \Rightarrow p_1 \leq p_3$$

$$E_{Attacker}(1,1) \leq E_{Attacker}(2,0) \Rightarrow p_1 + p_4 \leq p_1 + p_2 \Rightarrow p_4 \leq p_2$$

To summarize, a mixed strategy profile is a MSNE if: *Attacker* plays $(0,2)$, $(2,0)$ with probability 0.5 on each, and $Defender$ plays p_1, p_2, p_3, p_4 such that:

$$p_1 + p_2 = p_3 + p_4 = 0.5$$

$$p_1 \leq p_3, p_4 \leq p_2$$