1 Solutions to Homework 4

1.1 Q1

Let A be the event that the contestant chooses the door holding the car, and B be the event that the host opens a door holding a goat. $\neg A$ is the event that the contestant does *not* choose the door holding the car. If $P(A|B) < \frac{1}{2}$, then the contestant should switch; if $P(A|B) = \frac{1}{2}$, the contestant is indifferent, and if $P(A|B) > \frac{1}{2}$, the contestant should not switch. According to Bayes' Rule,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

In this problem, $P(A) = \frac{1}{3}, P(\neg A) = \frac{2}{3}$.

- Suppose the host always reveals a goat, regardless of whether the contestant chose the door with the car or not. Then $P(B|A) = 1, P(B|\neg A) = 1$, and $P(A|B) = \frac{1}{3}$, so the contestant should switch.
- Suppose the host opens a door randomly. Then $P(B|A) = 1, P(B|\neg A) = \frac{1}{2}$, and $P(A|B) = \frac{1}{2}$, so the contestant is indifferent.
- Suppose the host only reveals a goat when the contestant picks a goat. Then $P(B|A) = 0, P(B|\neg A) = 1$, and $P(A|B) = 0$, so the contestant should switch (it is certain that the car is behind the other door).
- Suppose the host only reveals a goat when the contestant picks a car. Then $P(B|A) = 1, P(B|\neg A) = 0$, and $P(A|B) = 1$, so the contestant should not switch (it is certain that the car is behind the contestant's door).

1.2 Q2

There are two information sets for Player 1, and one information set for Player 2, each with 2 actions. Therefore, Player 1 has 4 pure strategies, and Player 2 has 2 pure strategies.

1. If $p = 0$:

	A	F
OO	0,2	0,2
OE	-2,0	-1,1
EO	0,2	0,2
EE	-2,0	-1,1

2. If $p = 1$:

	A	F
OO	0,2	0,2
OE	0,2	0,2
EO	-1,-1	1,1
EE	-1,-1	1,1

3. If $p = 0.5$, we can take the weighted combination of the first two tables.

	A	F
OO	0,2	0,2
OE	-1,1	-0.5,1.5
EO	-0.5,0.5	-0.5, 1.5
EE	-1.5,-0.5	0,1

1.3 Q3

1. Player 1's information sets are: $\{H\}, \{L\}$. Player 2's information sets are: $\{HU, LU\}$ and $\{HD, LD\}$.
2. $P(HU) = pa, P(LU) = (1-p)b, P(HD) = p(1-a), P(LD) = (1-p)(1-b)$. Therefore, $P(\{HU, LU\}) = P(HU) + P(LU) = pa + (1-p)b$ and $P(\{HD, LD\}) = P(HD) + P(LD) = p(1-a) + (1-p)(1-b)$.
3. If Player 2's information sets are reached with positive probability, his beliefs should be:
   - at $\{HU, LU\}$: $\left(\frac{pa}{pa+(1-p)b}, \frac{(1-p)b}{pa+(1-p)b}\right)$
   - at $\{HD, LD\}$: $\left(\frac{p(1-a)}{p(1-a)+(1-p)(1-b)}, \frac{(1-p)(1-b)}{p(1-a)+(1-p)(1-b)}\right)$

   If either information set is not reached with positive probability, any beliefs are consistent.
4. We assume both players use pure strategies (therefore a, b must be 0 or 1). Player 1's pure strategies are: UU, UD, DU, DD, where the first letter is the action if Nature chooses H. Player 2's pure strategies are: $(M|U, M|D), (M|U, B|D), (B|U, M|D), (B|U, B|D)$ where $(M|U, M|D)$ means "if U occurs, choose M. if D occurs, choose M".
5. HUM, HUB, LUM, LUB.
6. Using $p = 1/4$, the players' expected payoffs are shown in this matrix:

	$(M\|U, M\|D)$	$(M\|U, B\|D)$	$(B\|U, M\|D)$	$(B\|U, B\|D)$
UU	10,2.5	10,2.5	6,3.5	6,3.5
UD	6.25,2.5	3.25,4.75	5.25,1.25	2.25,3.5
DU	9.5,2.5	8.5,1.25	6.5,4.75	4.5,3.5
DD	5.75,2.5	1.75,3.5	5.75,2.5	1.75,3.5

7. The NE are $(UU, (B|U, B|D))$ and $(DU, (B|U, M|D))$.

8. Let $(p_U, 1 - p_U)$ be Player 2's beliefs over $\{HU, LU\}$ and let $(p_D, 1 - p_D)$ be Player 2's beliefs over $\{HD, LD\}$. Then:

   - If NE $(DU, (B|U, M|D))$ is played, then $p_U = 0, p_D = 1$. We must also check that Player 2's actions are optimal given his beliefs; this is easily seen in this case since these beliefs are consistent with a pure strategy and Player 2 is playing his best response to pure strategies.

   - If NE $(UU, (B|U, B|D))$ is played, Player 1 does not play D, so at that information set, any beliefs are consistent. At the information set following U, Player 2's belief matches Nature's distribution: $p_U = p = \frac{1}{4}$. We want to check that Player 2's optimal actions, given beliefs, match the NE. This is simple for the information set following U; Player 2 is using his best response. For the information set following D, we compute the expected payoff to M and B:

   $$E_2(M|D) = 10p_D + 0(1 - p_D) = 10p_D, E_2(B|D) = 5p_D + 3(1 - p_D) = 3 + 2p_D$$

     We want to find beliefs that would justify choosing pure B. This is optimal if $E_2(B|D) > E_2(M|D)$, or if $p_D \leq \frac{3}{8}$.

   Therefore, the set of weak sequential equilibria are:

   - Strategies: $(DU, (B|U, M|D))$, Player 2's beliefs: $p_U = 0, p_D = 1$
   - Strategies: $(UU, (B|U, B|D))$, Player 2's beliefs: $p_U = \frac{1}{4}, 0 \leq p_D \leq \frac{3}{8}$

9. $(UU, (B|U, B|D))$ is a pooling equilibrium and $(DU, (B|U, M|D))$ is a separating equilibrium.