1 Solutions to Homework 5

	C	D
C	2,2	0,3
D	3,0	1,1

For the next two questions we will assume the Prisoner's Dilemma is played with the above payoffs.

1.1 Q1

1. "Grim Trigger" vs. "Grim Trigger"
   - (C, C), (C, C) (C, C) (C, C) (C, C)
   - Payoff sequence: $(2, 2), (2, 2), ...$
   - Both players receive a discounted average of 2.
2. "Tit-for-Tat" vs. "Always Defect"
   - (C, D), (D, D), (D, D), (D, D), (D, D)
   - Payoff sequence: $(0, 3), (1, 1), (1, 1), ...$
   - Player 1's discounted average is a constant sequence of 1 shifted 1 period forward, which is equal to δ. Player 2's discounted average is $(1 - \delta)3 + \delta = 3 - 2\delta$.
3. "Tit-for-Tat-D" vs. "Tit-for-Tat"
   - (D, C), (C, D), (D, C), (C, D), (D, C)
   - Payoff sequence: $(3, 0), (0, 3), (3, 0), (0, 3), ...$
   - Player 1's discounted average is $(1 - \delta)3(1 + \delta^2 + \delta^4 + ...) = \frac{3(1-\delta)}{1-\delta^2}$. Player 2's discounted average is $(1 - \delta)3\delta(1 + \delta^2 + \delta^4 + ...) = \frac{3\delta(1-\delta)}{1-\delta^2}$.

1.2 Q2

1. "Tit-for-Tat" vs. "Tit-for-Tat"
   - (D, C), (C, D), (D, C), (C, D), (D, C)
   - Payoff sequence: $(3, 0), (0, 3), (3, 0), (0, 3), ...$
   - Player 1's discounted average is $\frac{3(1-\delta)}{1-\delta^2}$. Payer 2's discounted average is $\frac{3\delta(1-\delta)}{1-\delta^2}$.
2. "Modified Grim Trigger" vs. "Modified Grim Trigger"
   - (D, C), (D, D), (D, D), (D, D), (D, D)

- Payoff sequence: $(3,0),(1,1),(1,1),...$
- Player 1's discounted average is $3-2\delta$. Player 2's discounted average is δ.

3. "Tit-for-Tat-D" vs. "Tit-for-Tat-D"

- (C,D), (D,C), (C,D), (D,C), (C,D)
- Payoff sequence: $(0,3),(3,0),(0,3),(3,0)...$
- Player 1's discounted average is $(1-\delta)3\delta(1+\delta^2+\delta^4+...)=\frac{3\delta(1-\delta)}{1-\delta^2}$. Player 2's discounted average is $(1-\delta)3(1+\delta^2+\delta^4+...)=\frac{3(1-\delta)}{1-\delta^2}$.

1.3 442.1 (Deviations from grim trigger strategy)

- If player 1 adheres to the strategy, she subsequently chooses D (because player 2 chose D in the first period). Player 2 chooses C in the first period of the subgame (player 1 chose C in the first period of the game), and then chooses D (because player 1 chooses D in the first period of the subgame). Thus the sequence of outcomes in the subgame is $((D,C),(D,D),(D,D),\ldots)$, yielding player 1 a discounted average payoff in the subgame of

$$(1-\delta)(3+\delta+\delta^2+\delta^3+\cdots)=(1-\delta)\left(3+\frac{\delta}{1-\delta}\right)=3-2\delta.$$

- If player 1 refrains from punishing player 2 for her lapse, and simply chooses C in every subsequent period, then the outcome in period 2 and subsequently is (C,C), so that the sequence of outcomes in the subgame yields player 1 a discounted average payoff of 2.

If $\delta > \frac{1}{2}$ then $2 > 3-2\delta$, so that player 1 prefers to ignore player 2's deviation rather than to adhere to her strategy and punish player 2 by choosing D. (Note that the theory does not consider the possibility that player 1 takes player 2's play of D as a signal that she is using a strategy different from the grim trigger strategy.)

1.4 445.1 (Tit-for-tat as a subgame perfect equilibrium)

Suppose that player 2 adheres to *tit-for-tat*. Consider player 1's behavior in subgames following histories that end in each of the following outcomes.

(C, C) If player 1 adheres to *tit-for-tat* the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is x. If she chooses D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (D, C) and (C, D), and her discounted average payoff is $y/(1+\delta)$. Thus we need $x \geq y/(1+\delta)$, or $\delta \geq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(C, D) If player 1 adheres to *tit-for-tat* the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $y/(1+\delta)$. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (C, C) in every period, and her discounted average payoff is x. Thus we need $y/(1+\delta) \geq x$, or $\delta \leq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, C) If player 1 adheres to *tit-for-tat* the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is $\delta y/(1+\delta)$. If she deviates to D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (D, D) in every period, and her discounted average payoff is 1. Thus we need $\delta y/(1+\delta) \geq 1$, or $\delta \geq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, D) If player 1 adheres to *tit-for-tat* the outcome is (D, D) in every period, so that her discounted average payoff is 1. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $\delta y/(1+\delta)$. Thus we need $1 \geq \delta y/(1+\delta)$, or $\delta \leq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

The same arguments apply to deviations by player 2, so we conclude that (*tit-for-tat*, *tit-for-tat*) is a subgame perfect equilibrium if and only if $\delta = (y-x)/x$ and $\delta = 1/(y-1)$, or $y - x = 1$ and $\delta = 1/x$.