A strategic game is a model of a situation with many interacting decision-makers.
A *strategic game* is a model of a situation with many interacting decision-makers.

A game has three parts:

1. **Players** (the decision-makers)
2. For each player, a set of **actions**. An **action profile** is a list of everyone's chosen action.
3. For each player, **preferences** over the set of action profiles (usually represented by a **payoff function**).
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The Prisoner’s Dilemma

- Players: two suspects to a crime, held by the police

- Preferences:
  - Suspect 1: \((F, Q) > (Q, Q) > (F, F) > (F, Q)\)
  - Suspect 2: \((Q, F) > (Q, Q) > (F, F) > (Q, F)\)

These preferences can be represented by payoff functions:

- Suspect 1: 
  - \(u_1(F, Q) = 3\)
  - \(u_1(Q, Q) = 2\)
  - \(u_1(F, F) = 1\)
  - \(u_1(Q, F) = 0\)

- Suspect 2: 
  - \(u_2(F, Q) = 0\)
  - \(u_2(Q, Q) = 2\)
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We can collect the payoff values into a **payoff matrix**:
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The two rows are the two possible actions of Player 1.
The two columns are the two possible actions of Player 2.

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<tr>
<th></th>
<th>Q</th>
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<tbody>
<tr>
<td>Q</td>
<td>2,2</td>
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- The two columns are the two possible actions of Player 2.
- In each cell, the first number is the payoff of Player 1; the second is the payoff of Player 2.
Let’s Play the Prisoner’s Dilemma

- Everyone should have two cards: one Black and one Red card.

How to play:

Start with two players, each with a Black and Red card.

Each player chooses to play Black or Red, and puts the card facedown.

Reveal both cards at the same time (why?)

Suppose you are Player 1. If you play Red, then you get +2 and Player 2 gets +0.

If you play Black, you get +0 and the other player gets +3.

So, Red is beneficial to you, while Black benefits the other player.

<table>
<thead>
<tr>
<th></th>
<th>Black</th>
<th>Red</th>
</tr>
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<tbody>
<tr>
<td>Black</td>
<td>3,3</td>
<td>0,5</td>
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Prof. Ronaldo CARPIO
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\[
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 & \text{Black} & \text{Red} \\
\hline
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\text{Red} & 5,0 & 2,2 \\
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This game has the same structure as Prisoner’s Dilemma.
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- **Black** corresponds to *Quiet*. These actions are sometimes named *Cooperate*, because they increase payoffs for the *other* player.
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Suppose you are working with a friend on a joint project.
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Project would be better if both work hard, but not worth the extra effort.
Suppose you are working with a friend on a joint project. Each of you can choose to *Work hard* or *Goof off* (be lazy). If the other person *Works hard*, each of you prefers to *Goof off*. Project would be better if both work hard, but not worth the extra effort.

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<th></th>
<th>Player 1</th>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Goof off</strong></td>
<td></td>
<td><strong>Work hard</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Work hard</strong></td>
<td>2,2</td>
<td>0,3</td>
<td></td>
</tr>
<tr>
<td><strong>Goof off</strong></td>
<td>3,0</td>
<td>1,1</td>
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Duopoly

- Two firms produce the same good.
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<td>-200, 1200</td>
</tr>
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Similarities to Prisoner’s Dilemma

- Names of actions and payoffs are different, but *relative* payoffs are the same.
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- Preferences (i.e. ranking) over outcomes are the same as in Prisoner’s Dilemma
- If both players cooperate, both get an outcome with good payoffs
- But if only one player chooses to defect, he gets an even better payoff (and cooperating player gets low payoff)
Applications of Prisoner’s Dilemma

- Arms Race
  - Players: Countries
  - Actions: Arm, Disarm
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- **Provision of a Public Good**
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- **Managing a Common Resource (Tragedy of the Commons)**
  - Players: Animal Herders
  - Actions: Reduce Grazing, Overgraze
Bach or Stravinsky? (also known as Battle of the Sexes)

- Two people want to go to a concert by either Bach or Stravinsky.
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<th>Stravinsky</th>
</tr>
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<tbody>
<tr>
<td>Bach</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>
Let’s Play Bach or Stravinsky

<table>
<thead>
<tr>
<th>Black (Bach)</th>
<th>Red (Stravinsky)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>0, 0</td>
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Matching Pennies

- Prisoner’s Dilemma and BoS have both conflict and cooperation. Matching Pennies is purely conflict.
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- Each of two people chooses either *Head* or *Tail*. 

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<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
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- If they are the same, Player 2 pays Player 1 $1.
- Each person cares only about the money he receives.

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Prof. Ronaldo CARPIO
CUR 412: Game Theory and its Applications, Lecture 2
Solution Concept

- We’ve defined the game. What outcomes are more likely to occur?
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- An assumption about the *behavior* of the players. We will assume rational behavior, i.e. choosing the action with the highest payoff.
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- A solution concept has two parts:
  - An assumption about the *behavior* of the players. We will assume rational behavior, i.e. choosing the action with the highest payoff
  - An assumption about the *beliefs* of the players.
Suppose you are Player 1. In order to choose the best action, you need to have some idea of what Player 2 will choose.
Beliefs

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- This is called a belief about Player 2. Includes: the rules of the game, Player 2’s payoff function, but also...

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- Reasoning about what other players know (and what they know you know...) is called higher-order knowledge.
- We’ll make a (very strong!) simplifying assumption: beliefs of all players are correct.
- subscript $i$ denotes player $i$ or an action of player $i$

Action profile (i.e. a list of all actions chosen by all players) $a^*$ is composed of $a^*_i$ and $a^*_{-i}$:

$$a^* = (a^*_i, a^*_{-i})$$

$a^*_i$ is the action chosen by player $i$

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Nash Equilibrium

- This solution concept assumes that:
  - Players are rational (i.e. choose the highest payoff), given beliefs about other players
  - Beliefs of all players are correct
  - We want to find an outcome that is a steady state, that is, starting from that outcome, no player wants to deviate.

Definition: The action profile $a^*$ in a strategic game is a Nash Equilibrium if, for every player $i$ and every action $b_i$ of player $i$, $a^*$ is at least as preferable for player $i$ as the action profile $(b_i, a^*_{-i})$:

$$u_i(a^*) \geq u_i(b_i, a^*_{-i})$$ for every action $b_i$ of player $i$. 
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$$u_i(a^*) \geq u_i(b_i, a^*_{-i})$$
Nash Equilibrium

- This solution concept assumes that:
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\[
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Note that this definition does not guarantee that a game has a Nash equilibrium.
Nash Equilibrium

- Note that this definition does not guarantee that a game has a Nash equilibrium.
- Some games may have one, more than one, or zero Nash equilibria.
Prisoner’s Dilemma

\[
\begin{array}{c|cc}
 & Q & F \\
\hline
Q & 2,2 & 0,3 \\
F & 3,0 & 1,1 \\
\end{array}
\]

- \((F, F)\) is the unique Nash equilibrium. No other action profile satisfies the conditions:

- Joint project: both will Goof off
- Duopoly: both will charge a Low price (this is bad for the firms, but good for consumers)
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Prisoner’s Dilemma

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- Note that \( F \) is the best action for each player, *regardless* of what the other player does. (This is not the case in other games).

Individual rationality can lead to a socially inefficient outcome. How might players reach the better outcome, while still behaving rationally (payoff-maximizing)?

Need to change the structure of the game, e.g.:
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Both outcomes are compatible with a steady state.
Matching Pennies

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- There is *no* Nash equilibrium.
- For every action profile, at least one player has an incentive to deviate.
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In a Nash equilibrium, each player’s equilibrium action has to be \textit{at least as good} as every other action, not necessarily better.
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Consider the following game:

\[
\begin{array}{ccc}
T & L & M & R \\
T & 1,1 & 1,0 & 0,1 \\
B & 1,0 & 0,1 & 1,0 \\
\end{array}
\]

\((T, L)\) is the unique Nash equilibrium.

However, when Player 2 plays \(L\), Player 1 is indifferent between \(T\) and \(B\).

This is called a \textit{non-strict} or \textit{weak} Nash equilibrium.
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(T, L) is the unique Nash equilibrium.

However, when Player 2 plays L, Player 1 is indifferent between T and B.

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Definition: The action profile $a^*$ in a strategic game is a strict Nash equilibrium if, for every player $i$ and every action $b_i \neq a_i^*$ of player $i$, $a^*$ is strictly preferred by player $i$ to the action profile $(b_i, a_{-i}^*)$:

$$u_i(a^*) > u_i(b_i, a_{-i}^*)$$

for every action $b_i \neq a_i^*$ of player $i$. 
Suppose that the players other than Player $i$ play the action list $a_{-i}$.

Let $B_i(a_{-i})$ be the set of Player $i$'s best (i.e. payoff-maximizing) actions, given that the other players play $a_{-i}$.

There may be more than one.

$B_i$ is called the best response function of Player $i$.

$B_i$ is a set-valued function, that is, it may give a result with more than one element.

Every member of $B_i(a_{-i})$ is a best response of Player $i$ to $a_{-i}$.
Best Response Functions

- Suppose that the players other than Player $i$ play the action list $a_{-i}$.
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Prisoner’s Dilemma

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- $B_i(Q) = \{ F \}$ for $i = 1, 2$
Prisoner’s Dilemma

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\begin{array}{c|cc}
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- \( B_i(F) = \{ F \} \) for \( i = 1, 2 \)
\[ \begin{array}{cc}
\text{Bach} & \text{Stravinsky} \\
\hline
\text{Bach} & 2, 1 & 0, 0 \\
\text{Stravinsky} & 0, 0 & 1, 2 \\
\end{array} \]

- \( B_i(Bach) = \{Bach\} \) for \( i = 1, 2 \)
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- \( B_1(M) = \{T\} \)
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- \(B_2(B) = \{M\}\)
Using Best Response Functions to find Nash Eq.

- **Proposition**: The action profile \( a^* \) is a Nash equilibrium if and only if every player’s action is a best response to the other players’ actions:

\[
\begin{align*}
    a^*_i & \in B_i(a^*_{-i}) \\
    a^*_i = b_i(a^*_{-i})
\end{align*}
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- **Proposition**: The action profile $a^*$ is a Nash equilibrium if and only if every player’s action is a best response to the other players’ actions:

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- If the best-response function is *single-valued*:

  $$ a_i^* = b_i(a_{-i}^*) $$  

If the best-response function is 2 players, condition 1 is equivalent to:

$$ a_1^* = b_1(a_2^*) $$
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Using Best Response Functions to find Nash Eq.

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Using Best Response Functions to find Nash Eq.

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Finding Nash equilibrium with Best-Response functions

- We can use this to find Nash equilibria when the action space is continuous.
Finding Nash equilibrium with Best-Response functions

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- Step 1: Calculate the best-response functions.
Finding Nash equilibrium with Best-Response functions

- We can use this to find Nash equilibria when the action space is continuous.
- Step 1: Calculate the best-response functions.
- Step 2: Find an action profile $a^*$ that satisfies:
  \[
  a_i^* \in B_i(a_{-i}^*) \quad \text{for every player } i
  \]
We can use this to find Nash equilibria when the action space is continuous.

- Step 1: Calculate the best-response functions.
- Step 2: Find an action profile \( a^* \) that satisfies:

\[
a_i^* \in B_i(a_{-i}^*) \quad \text{for every player } i
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- Or, if every player's best-response function is single-valued, find a solution of the \( n \) equations (\( n \) is the number of players):
Finding Nash equilibrium with Best-Response functions

- We can use this to find Nash equilibria when the action space is continuous.
- Step 1: Calculate the best-response functions.
- Step 2: Find an action profile $a^*$ that satisfies:
  \[ a_i^* \in B_i(a^*_{-i}) \quad \text{for every player } i \]

- Or, if every player’s best-response function is single-valued, find a solution of the $n$ equations ($n$ is the number of players):
  \[ a_i^* = b_i(a^*_{-i}) \quad \text{for every player } i \]
Example: synergistic relationship (37.2 in book)

- Two individuals.

\[ \text{Payoff to Player } i: u_i(a_i) = a_i \cdot (c + a_j - a_i), \text{ where } c > 0 \]
Example: synergistic relationship (37.2 in book)

- Two individuals.
- Each decides how much effort to devote to relationship.

\[
\begin{align*}
    \text{Amount of effort } a_i \text{ is a non-negative real number (so the action space is infinite)} \\
    \text{Payoff to Player } i: u_i(a_i) &= a_i \cdot (c + a_j - a_i), \text{ where } c > 0 \text{ is a constant.}
\end{align*}
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Example: synergistic relationship (37.2 in book)

- Two individuals.
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Amount of effort $a_i$ is a non-negative real number (so the action space is infinite)
Payoff to Player $i$: $u_i(a_i) = a_i \cdot (c + a_j - a_i)$, where $c > 0$ is a constant.
Finding the Nash Equilibrium

- Construct players’ best-response functions:

\[ u_i(a_i) = a_i \cdot (c + a_j - a_i) \]

Given \( a_j \), this becomes a quadratic:

\[ u_i(a_i) = a_i \cdot c + a_i \cdot a_j - a_i^2 \]

Best response to \( a_j \) is when this quadratic is maximized. Take the derivative and set to 0.

\[ c + a_j - 2a_i = 0 \]

\[ a_i = c + a_j \]

So, best response functions are:

\[ b_1(a_2) = c + a_2 \]

\[ b_2(a_1) = c + a_1 \]
Finding the Nash Equilibrium

- Construct players’ best-response functions:
- Player $i$’s payoff function: $u_i(a_i) = a_i \cdot (c + a_j - a_i)$
Finding the Nash Equilibrium

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\[ \Rightarrow a_i = \frac{c + a_j}{2} \]
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Finding the Nash Equilibrium

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  - Player $i$’s payoff function: $u_i(a_i) = a_i \cdot (c + a_j - a_i)$
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c + a_j - 2a_i = 0
\]

\[
\rightarrow a_i = \frac{c + a_j}{2}
\]

- So, best response functions are:
  - $b_1(a_2) = \frac{c + a_2}{2}$
  - $b_2(a_1) = \frac{c + a_1}{2}$
The pair \((a_1, a_2)\) is a Nash equilibrium if \(a_1 = b_1(a_2)\) and \(a_2 = b_2(a_1)\).
Finding the Nash Equilibrium

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- Solving the two equations...
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- gives a unique solution \((c, c)\).
Finding the Nash Equilibrium

- The pair \((a_1, a_2)\) is a Nash equilibrium if \(a_1 = b_1(a_2)\) and \(a_2 = b_2(a_1)\).
- Solving the two equations

\[
\begin{align*}
    a_1 &= \frac{c + a_2}{2} \\
    a_2 &= \frac{c + a_1}{2}
\end{align*}
\]

- gives a unique solution \((c, c)\).
- Therefore, this game has a unique Nash equilibrium:
\(a_1 = c, a_2 = c\).
The intersection of \( b_1(a_2) = \frac{c + a_2}{2} \) and \( b_2(a_1) = \frac{c + a_1}{2} \) is the Nash equilibrium.
A player's action is *strictly dominated* by another action if it gives a lower payoff, regardless of what other players do.

**Definition:** Player \( i \)'s action \( b_i \) strictly dominates action \( b'_i \) if
\[
u_i(b_i, a_{-i}) > u_i(b'_i, a_{-i})
\]
for every \( a_{-i} \).

We say action \( b'_i \) is strictly dominated.

A strictly dominated action cannot be a best response to any actions of the other players, because some other action exists that gives a higher payoff.

Therefore, a strictly dominated action is not played in any Nash equilibrium.
Dominated Actions

- A player's action is *strictly dominated* by another action if it gives a lower payoff, regardless of what other players do.

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- Therefore, a strictly dominated action is not played in any Nash equilibrium.
For both players, $F$ strictly dominates $Q$: regardless of the other player’s action, $F$ gives a higher payoff.
Prisoner’s Dilemma

For both players, $F$ strictly dominates $Q$: regardless of the other player’s action, $F$ gives a higher payoff.

This eliminates all outcomes where $Q$ is played as Nash equilibria.
Neither action is strictly dominated.
Next Week

- Please read the rest of Chapter 2, and Chapter 3.1-3.3.
- If you don’t have the textbook, check the course website for information on how to get the textbook chapters.