0.1 Chapter 10: Extensive Games with Imperfect Information

Up to now, the extensive games we have studied have been games with *perfect information*, i.e. players always know exactly what has happened in the past when making decisions. This eliminates any kind of uncertainty about the past, e.g:

- What if moves by another player are unknown (as in simultaneous-move games)?
- What if some parameter of the game or the players depends on a random variable?

We can allow for uncertainty in this way: if information about the past is hidden from a player when he makes his decision, we’ll say the player *cannot distinguish* between histories (decision nodes) that differ based on that information.

Consider the simultaneous-move game BoS. We cannot model this as an extensive game with perfect information, because each player must not know what the other player chose. With imperfect information, we can model the game this way: Assume Player 1 moves first, choosing $B$ or $S$, but this information is hidden from Player 2. Player 2 cannot distinguish between the history where Player 1 chose $B$ and where he chose $S$. On the tree diagram, we denote this with a dashed line connecting the two nodes. We call this group of nodes (i.e. a group of histories) an *information set* of Player 2.

The actions available to a player at all histories (nodes) in an information set must be the same. Note that we could also represent BoS as an extensive game where Player 2 moves first, and Player 1’s information set contains $B$ and $S$. 

With imperfect information, our definition of subgame must be modified: if a subgame contains a node in an information set, it must contain all the nodes in that information set (i.e. it makes no sense to have a game where not all the actions are specified). The extensive form of BoS has only one subgame, since we cannot split Player 2’s information set.

We can convert extensive games with imperfect information into strategic form, just as before; we can also allow for pure and mixed strategies in the strategic form. Now, a pure strategy must specify an action at every information set; a mixed strategy specifies a probability distribution over actions at every information set. In the extensive form of BoS, both players have a single information set, at which the set of possible actions is $B$ and $S$; so a strategy would specify a single distribution over $B, S$.

Many interesting situations involve uncertainty due to a random outcome in the past. In Chapter 7, we saw how to incorporate randomness by adding a player called ”Chance” or ”Nature”, who chooses actions randomly, according to a known probability distribution. Imperfect information is a natural way to handle randomness in games. Note that with randomness, we must now assume preferences over lotteries; specifically, we will assume von Neumann-Morgenstern utilities (i.e. lotteries are valued based on expected payoff).

Consider this simple two-player card game:

1. Both players add $1 to the pot (i.e. the prize money).
2. Player 1 draws a card that can be either High or Low, with probability $\frac{1}{2}$.
3. Player 1 can choose to See (i.e. resolve the game now based on his cards), or Raise (increase the stakes). If See is chosen, the card is revealed:
   - If High, Player 1 wins the prize, for a net payoff of 1. Player 2’s payoff is -1. We denote this as $(1, -1)$
   - If Low, Player 2 wins the prize. Payoffs are $(-1, 1)$

   If Raise is chosen, Player 1 adds $1 to the prize, and it becomes Player 2’s turn.
4. Player 2 can choose to Pass (give up) or Meet (i.e. match the increase in stakes and stay in the game).
   - If Pass is chosen, Player 1 wins the prize. Payoffs are $(1, -1)$.
   - If Meet is chosen, Player 2 adds $1 to the prize; then the game is resolved based on the cards held.
     - If Player 1’s card is High, payoffs are $(2, -2)$.
     - If Player 1’s card is Low, payoffs are $(-2, 2)$. 
Here, Player 1’s goals depend on which card he draws. If Player 1 draws High, it is plausible to think that Player 1 would want to Raise more often, since it is possible to get a high payoff of 2; likewise, if Player 1 draws Low, he may want to Raise less often, to avoid the low payoff of -2. Knowing the probability distribution of High vs. Low is important for Player 2 in deciding his optimal strategy. If Player 2 knows the probability of High is large, he would want to choose Pass more often, and vice versa. Later on, we will see how to exactly compute Player 2’s optimal strategy based on the probability of High vs. Low.

First, let’s convert the game into strategic form and find the Nash equilibrium, for both pure and mixed strategies. Player 1 has 2 information sets, after the histories High and Low. Player 1’s pure strategies are (See,See), (See,Raise), (Raise,See), and (Raise,Raise). Player 2 has 1 information set, so Player 2’s pure strategies consist of one action, which may be Pass or Meet. We assume the probability of High, Low occurring is 1/2 for each. We can calculate the expected payoffs for each pure strategy profile, shown in Figure 320.1 of the textbook.

<table>
<thead>
<tr>
<th></th>
<th>Pass</th>
<th>Meet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise, Raise</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Raise, See</td>
<td>0, 0</td>
<td>0.5, -0.5</td>
</tr>
<tr>
<td>See, Raise</td>
<td>1, -1</td>
<td>-0.5, 0.5</td>
</tr>
<tr>
<td>See, See</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

By inspecting the table, we can see there is no pure strategy Nash equilibrium (this is also suggested by the fact that this is a constant-sum game). Now, let’s look at mixed strategies. We know that a mixed strategy always exists in a finite game. Let’s examine the pure strategies to see if they could be part of a mixed strategy NE:

- Player 1’s best response to Pass is a combination of (See, Raise) and (Raise, Raise).
- Player 1’s best response to Meet is (Raise, See).

Can Player 2 play a pure strategy in MSNE?
• Suppose Player 2 plays Pass. Player 1’s best response is a combination of (See, Raise) and (Raise, Raise). Player 2’s best response to that is Meet. So pure Pass cannot be played in a MSNE.

• Suppose Player 2 plays Meet. Player 1’s best response is (Raise, See), and Player 2’s best response to that is Pass. So pure Meet cannot be played in a MSNE.

This tells us that Player 2 will not play a pure strategy in MSNE. Next, let’s look at Player 1.

• (See, See) is strictly dominated by a combination of (Raise, See) and (Raise, Raise). Therefore, it won’t be played in MSNE.

• (See, Raise) is weakly dominated by (Raise, Raise). For any mixed strategy of Player 2 that places positive probability on Meet, (See, Raise) is not a best response; Player 1 can do better by allocating more probability to (Raise, Raise). Therefore, (See, Raise) won’t be played in MSNE.

• Player 2’s best response to (Raise, See) is Pass, and Player 1’s best response to that is a combination of (See, Raise) and (Raise, Raise). Therefore, pure (Raise, See) won’t be played in MSNE.

• Player 2’s best response to (Raise, Raise) is Meet, and Player 1’s best response to that is (Raise, See). Therefore, pure (Raise, Raise) won’t be played in MSNE.

This tells us that Player 1’s strategy is a combination of (Raise, See) and (Raise, Raise).

We can now use the condition that in a MSNE, expected payoffs to actions with positive probability must be equal.

Suppose Player 1 puts probability \( p \) on (Raise, Raise), with \( (1 - p) \) on (Raise, See). Player 2 places probability \( q \) on Pass, and \( (1 - q) \) on Meet.

• For Player 1, \( E_1(Raise, Raise) = q, E_1(Raise, See) = \frac{1}{2}(1 - q) \). Setting these to be equal gives \( q = \frac{1}{3} \).

• For Player 2, \( E_2(Pass) = -p, E_2(Meet) = -\frac{1}{2}(1 - p) \). Setting these to be equal gives \( p = \frac{1}{3} \).

Therefore, there is a unique MSNE, with Player 1 playing (Raise, Raise) with \( \frac{1}{3} \) and (Raise, See) with \( \frac{2}{3} \), and Player 2 playing Pass with \( \frac{1}{3} \) and Meet with \( \frac{2}{3} \).

As before, the Nash equilibrium of the strategic form may include behavior that would be irrational when considered in a subgame (e.g. incredible threats). We would like to develop something similar to the notion of subgame perfect equilibrium, in which strategies must also be optimal at each stage of the game. In order to do this, we need to specify a player’s mixed strategy at each information set.

**Definition 324.2 (Behavioral strategy in extensive games):** A behavioral strategy of player \( i \) in an extensive game is a function that assigns to each of \( i \)’s information sets \( I_i \), a probability distribution over the actions available at \( I_i \), with the property that each probability distribution is independent of every other distribution. A behavioral strategy profile is a list of behavioral strategies for all players.
In the extensive form of BoS, each player has one information set, so a behavioral strategy specifies a probability distribution over actions in that information set. This has the same form as the mixed strategy of the strategic form. In the Card Game:

- Player 1 has two information sets, after High and Low, so a behavioral strategy consists of two probability distributions: one over See, Raise after High, and one after See, Raise after Low.
- Player 2 has one information set, so a behavioral strategy consists of one probability distribution over Pass, Meet.

We also need to find the optimal action(s) at each information set. However, if the optimal action depends on which history has occurred (as in the Card Game: the optimal action for Player 2 is different if High or Low has occurred), then we need some way to specify what Player 2 thinks has happened at the information set. We do this by modeling beliefs as a probability distribution.

### 0.2 Beliefs as a Probability Distribution

In the Card Game, consider Player 2’s information set. He knows that there are two possibilities for Player 1’s card: High and Low. Some possible beliefs of Player 2 are:

- High and Low are equally likely.
- High is more likely to have occurred than Low.
- It is impossible for Low to have occurred. High has occurred with certainty.

We can formulate these statements in a precise way with a probability distribution (this is sometimes called the subjectivist view of probability). Player 2 places a probability on High and Low that must add up to 1; the relative sizes of the probabilities reflects Player 2’s opinion on how likely it is that High vs. Low has occurred. For example:

- If Player 2’s opinion is that High and Low are equally likely, beliefs are modeled by a probability distribution \( \left( \frac{1}{2}, \frac{1}{2} \right) \).
- If Player 2’s opinion is that High is more likely to have occurred, the probability distribution is of the form \( (p, 1-p) \) where \( p > \frac{1}{2} \).
- If Player 2’s opinion is that Low is impossible, the probability distribution is \((1, 0)\).

### 0.3 Beliefs Conditional on a Behavioral Strategy

Suppose Player 1’s behavioral strategy is:

- After High, play Raise with probability \( p_H \), and See with probability \( 1 - p_H \).
- After Low, play Raise with probability \( p_L \), and See with probability \( 1 - p_L \).
On Player 2’s turn, he knows that Raise has occurred (since the game has not ended yet). What should his beliefs be on the occurrence of High vs. Low? We can compute these probabilities exactly, using Bayes’ Rule.

### 0.4 Conditional Beliefs and Bayes’ Rule

This is presented in the appendix (Chapter 17) of the textbook. Suppose $A$ and $B$ are two random events, with probability of occurrence $P(A), P(B)$ respectively. We denote the event that both $A$ and $B$ occur, $A \cap B$ (think of this as the intersection of possible worlds where $A$ occurs, and where $B$ occurs). If $A$ and $B$ are independent, then $P(A \cap B) = P(A)P(B)$.

The conditional probability of an event $A$ given an event $B$ is the probability of $A$ occurring, if we already know that $B$ has occurred. This is denoted as $P(A|B)$. If $A$ and $B$ are independent, then $B$ occurring does not tell us anything about whether $A$ is more likely to occur, so $P(A|B) = P(A)$. $P(A)$ is the unconditional probability of $A$ occurring (i.e. the probability of $A$, knowing nothing else). It is also called the prior probability of $A$, because it is the belief that $A$ occurs prior to receiving any new information about the world. The conditional probability $P(A|B)$ is then called the posterior probability of $A$ after learning that $B$ has occurred.

Assuming that $P(B) > 0$, then the conditional probability of $A$ given $B$, $P(A|B)$, is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

It follows that $P(A|B)P(B) = P(A \cap B)$, i.e. the probability that both $A$ and $B$ occur is the joint probability that $B$ occurs, and that $A$ occurs once we know $B$ has occurred. Likewise, $P(B|A)P(A) = P(A \cap B)$. We can replace $P(A \cap B)$ in the right hand side above with $P(B|A)P(A)$, giving us

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This is known as Bayes’ Rule or Bayes’ Theorem. We will be studying games with binary (two possible outcomes) random variables, e.g. High and Low. Let $\neg A$ be the negation of $A$, i.e. the event that $A$ does not occur. For example, if $A$ is the event that the card is High, then $\neg A$ is the event that the card is not High, i.e. that it is Low. We can then decompose $P(B)$ into $P(B|A)P(A) + P(B|\neg A)P(\neg A)$.

In the Card Game, let $A$ be the event that Player 1’s card is High, and $\neg A$ be the event that Player 1’s card is Low, with probabilities $P(A), P(\neg A)$ which we have assumed to be $\frac{1}{2}$ each. Let $B$ be the event that Player 1 chooses Raise. Then $P(B|A)$ is the probability of Raise conditional on High, which we have assumed to be $p_H$ in Player 1’s behavioral strategy. $P(B|\neg A)$ is the probability of Raise conditional on Low, which is $p_L$. $P(B)$ is the unconditional probability of Player 1 choosing Raise, which is equal to $P(B|A)P(A) + P(B|\neg A)P(\neg A) = p_H \frac{1}{2} + p_L \frac{1}{2}$.

We want to find $P(A|B)$, the probability that High was drawn conditional on Raise. This
will give the correct beliefs for Player 2 at his information set. Using Bayes' Rule, we get:

\[
P(\text{High}|\text{Raise}) = \frac{p_H \frac{1}{2}}{p_H \frac{1}{2} + p_L \frac{1}{2}} = \frac{p_H}{p_H + p_L}
\]

\[
P(\text{Low}|\text{Raise}) = \frac{p_L \frac{1}{2}}{p_H \frac{1}{2} + p_L \frac{1}{2}} = \frac{p_L}{p_H + p_L}
\]

Now, with Player 2’s beliefs at his information set, we can calculate the expected payoff to Player 2 of his actions. Let’s state some definitions which will lead to our concept of equilibrium:

**Definition 324.1**: A **belief system** is a function that assigns to each information set, a probability distribution over the histories in that information set.

A belief system is simply a list of beliefs at every information set. In the Card Game, a belief system will specify two beliefs for Player 1, one for each information set (since there is only one history in each, the belief will simply be a probability of 1). For Player 2, there will be one belief at his information set, a probability distribution over High, Raise and Low, Raise.

**Definition 325.1**: An **assessment** is a pair consisting of a profile of behavioral strategies, and a belief system. An assessment is called a **weak sequential equilibrium** if these two conditions are satisfied:

- All players’ strategies are optimal when they have to move, given their beliefs and the other players’ strategies. This is called the **sequential rationality** condition.
- All players’ beliefs are consistent with the strategy profile, i.e. their beliefs are what is computed by Bayes’ Rule. This is called the **consistency of beliefs** condition.

When we were defining the Nash equilibrium solution concept, we also had the requirement that strategies were optimal given beliefs, and beliefs were correct. This was easy to do because there was no uncertainty.

Finally, let’s find the weak sequential equilibrium for the Card Game. As we showed above, if Player 1’s behavioral strategy is:

- After High, play Raise with probability \( p_H \) and See with probability \( 1 - p_H \).
- After Low, play See with probability \( p_L \) and See with probability \( 1 - p_L \).

Then the beliefs of Player 2 that are consistent with this behavioral strategy is \((\frac{p_H}{p_H + p_L}, \frac{p_L}{p_H + p_L})\).

Player 2’s expected payoffs, given these beliefs, are:

\[
E_2(\text{Pass}) = \frac{p_H}{p_H + p_L}(-1) + \frac{p_L}{p_H + p_L}(-1) = -1
\]

\[
E_2(\text{Meet}) = \frac{p_H}{p_H + p_L}(-2) + \frac{p_L}{p_H + p_L}2 = \frac{2p_L - 2p_H}{p_H + p_L}
\]

Therefore, Player 2 will choose the pure strategy Pass if \(-1 > \frac{2p_L - 2p_H}{p_H + p_L}\), or rearranging, if \(p_H > 3p_L\). If \(p_H < 3p_L\), Player 2 will choose pure Meet, and if \(p_H = 3p_L\), any mixed strategy will give the same expected payoff.
Now, consider Player 1’s decision. He takes Player 2’s response as given, and chooses \( p_H, p_L \) to maximize expected payoff. The range of Player 1’s choices can be divided into three areas:

\[
\begin{align*}
\text{Player 2 mixes between Pass, Meet} \\
p_H = 3p_L \\
\text{Player 2 chooses Pass} \\
\text{Player 2 chooses Meet} \\
p_H
\end{align*}
\]

In general, we would need to calculate Player 1’s expected payoff as a function of \( p_H, p_L \) for each region, then choose the highest one. However, we can take advantage of the fact that expected payoffs are linear in \( p_H, p_L \), and a linear function over a polygon is maximized and minimized on the corners. So we only need to check \((p_L, p_H) = (0, 0), (0, 1), (\frac{1}{3}, 1), (1, 0), \) and \((1, 1)\). Suppose Player 2 places probability \( q \) on Pass (which can be 0 or 1). Then Player 1’s expected payoff is:

\[
\frac{1}{2} \left( (1 - p_H) + p_H(q + 2(1 - q)) \right) + \frac{1}{2} \left( (1 - p_L) + p_L(q - 2(1 - q)) \right)
\]

\[
= \frac{1}{2} \left( p_H(1 - q) + p_L(3q - 1) \right)
\]

- If \( p_H > 3p_L \), then Player 2 chooses \( q = 1 \), and Player 1’s expected payoff at \((p_L, p_H) = (0, 1)\) is 0.
- If \( p_H = 3p_L \), then all values of \( q \) give the same expected payoff to Player 2. Player 1’s expected payoff at \((p_L, p_H) = (0, 0)\) is 0. At \((\frac{1}{3}, 1)\), it is \( \frac{1}{3} \).
- If \( p_H < 3p_L \), then Player 2 chooses \( q = 0 \), and Player 1’s expected payoff at \((1, 0)\) is \(-\frac{1}{2}\). At \((1, 1)\), it is 0.

Therefore, the optimal choice for Player 1 is \( p_H = 1, p_L = \frac{1}{3} \). Note that Player 1 sometimes chooses Raise on Low - this can be thought of as “bluffing”.
We can convert between a behavioral strategy and a mixed strategy of the strategic form. Suppose Player 1's mixed strategy is denoted $\alpha_1$, where $\alpha_1(Raise, Raise)$ is the probability placed on $(Raise, Raise)$, etc. Then the following relationships between $\alpha_1$ and $p_H, p_L$ must hold:

\[
\begin{align*}
P(Raise|High) &= p_H = \alpha_1(Raise, Raise) + \alpha_1(Raise, See) = 1 \\
P(See|High) &= 1 - p_H = \alpha_1(See, Raise) + \alpha_1(See, See) = 0 \\
P(Raise|Low) &= p_L = \alpha_1(Raise, Raise) + \alpha_1(See, Raise) = \frac{1}{3} \\
P(See|Low) &= 1 - p_L = \alpha_1(Raise, See) + \alpha_1(See, See) = \frac{2}{3}
\end{align*}
\]

The second equation tells us that $\alpha_1(See, Raise) = \alpha_1(See, See) = 0$, since probabilities must be non-negative. Therefore, $\alpha_1(Raise, Raise) = \frac{1}{3}$, and $\alpha_1(Raise, See) = \frac{2}{3}$, which is the same mixed strategy as in the MSNE found above.

Next week, we will find all parts of the weak sequential equilibrium for the Card Game, and we will begin studying signaling games.