0.1 Consistency of Beliefs with Strategies

In the previous lecture, we showed how to model beliefs at an information set as a probability distribution over the set of possible histories at that information set. We defined an equilibrium concept, weak sequential equilibrium, that had two parts: beliefs at every information set, and strategies for every information set, with the requirements that:

- Given beliefs and strategies of other players, each player’s strategy at each of his information sets should be optimal;
- Beliefs at every information set should be consistent with the strategies of all the players.

Let’s elaborate on what we mean that beliefs should be consistent with strategies. We would like players to have correct beliefs at their information sets; that is, their subjective probability distribution over possible histories should be the same as the true probabilities that would result if all players followed their strategies. For example, consider the Card Game:

Player 2 has an information set with two possible histories, High, Raise and Low, Raise. The true probability distribution over these two histories is given by Bayes’ Rule:

\[
P(\text{High} | \text{Raise}) = \frac{P(\text{Raise} | \text{High}) P(\text{High})}{P(\text{Raise})}
\]

\[
= \frac{P(\text{Raise} | \text{High}) P(\text{High})}{P(\text{Raise} | \text{High}) P(\text{High}) + P(\text{Raise} | -\text{High}) P(-\text{High})}
\]

However, this is undefined if \(P(\text{Raise}) = 0\). This is possible if the players’ strategies ensure that Player 2’s information set is never actually reached. For example, if Player 1’s strategy is
$High \rightarrow \text{See}, Low \rightarrow \text{See}$, then the game will always end before reaching Player 2’s information set, so $P(\text{Raise}) = 0$. In this case, any set of beliefs at Player 2’s information set would be consistent. We say that this strategy profile does not restrict the possible set of beliefs at Player 2’s information set.

### 0.2 Weak Sequential Equilibrium of the Card Game

Finally, we can state the weak sequential equilibrium of the card game. We need two parts: the beliefs, and the strategies.

- **Strategies**:
  - Player 1’s behavioral strategy is: $High \rightarrow (1,0); Low \rightarrow \left(\frac{1}{3}, \frac{2}{3}\right)$, where the first number in the pair is the probability of playing $\text{Raise}$.
  - Player 2’s behavioral strategy depends on Player 1’s behavioral strategy. Suppose Player 1’s behavioral strategy is specified by $p_H = P(\text{Raise}|High), p_L = P(\text{Raise}|Low)$. Then Player 2’s behavioral strategy is: $(1,0)$ if $p_H > 3p_L$; $(q,1-q), 0 \leq q \leq 1$ if $p_H = 3p_L$; $(0,1)$ if $p_H < 3p_L$, where the first number in the pair is the probability of playing $\text{Pass}$.

- **Beliefs**: Player 2’s beliefs are: $\left(\frac{p_H}{p_H + p_L}, \frac{p_L}{p_H + p_L}\right)$.

### 0.3 Signaling Games

Signaling games are used to model a situation that is frequently encountered in real life: when one player’s "type" is unobserved by the other players. For example, when you buy something in the supermarket, you do not know what the true quality level of the good is; when an employer hires a worker, he does not know what the true ability level of the worker is. In such situations, it may be possible for the player with the unknown type to demonstrate (i.e. signal) his type through his actions. Signaling games usually have the following four components:

1. Nature chooses a type for Player 1 that Player 2 cannot observe, but which matters for Player 2’s payoffs.
2. Player 1 can choose an action that has a different cost, depending on his type.
3. Player 1 chooses an action first, then Player 2 responds after observing Player 1’s choice (but not his type).
4. Player 2’s beliefs are updated by Player 1’s choice. Player 2 then makes his choice as a best response, given his updated beliefs.

*Separating* equilibria are equilibria in which it is possible for Player 2 to distinguish between types of Player 1 based on their actions (that is, each type of Player 1 finds it optimal to choose a different action). *Pooling* equilibria are those in which this is not possible, so different types of Player 1 choose the same action.
0.4 10.7: Education as a signal of ability

Why do students obtain a college degree? One reason is that the knowledge they gain in college will increase their skills and abilities. However, there is another possible reason: perhaps students use degrees to differentiate themselves from other students when applying for jobs. This can hold even if the degree itself does not increase ability. Here is a simple model to illustrate this phenomenon:

Suppose the ability level of a worker can be measured by a single number, and there are two types of workers: "high" and "low"-ability workers, denoted $H$ and $L$, with $L < H$. This type is known to the workers, but cannot be directly observed by employers. Workers can choose to obtain some amount of education, which has no effect on ability, but costs less for the $H$-type worker. The sequence of the game is as follows:

- Chance chooses the type of the worker at random; the probability of $H$ is $p$.
- The worker, who knows his type, chooses an amount of education $e \geq 0$. The cost of education is different according to type; for a $L$-type worker, the cost is $e/L$; for a $H$-type worker, it is $e/H$.
- Two firms observe the worker’s choice of $e$ (but not his type), and simultaneously offer two wages, $w_1$ and $w_2$.
- The worker chooses one of the wage offers and works for that firm. The worker’s payoff is his wage minus the cost of education. The firm that hires the worker gets a payoff of the worker’s ability, minus the wage. The other firm gets a payoff of 0.

This game is shown in this diagram:
We claim there is a weak sequential equilibrium in which a $H$-type worker chooses a positive amount of education, and a $L$-type worker chooses zero education. Consider this assessment (i.e. beliefs plus strategies), where $e^*$ is a positive number (to be determined):

- **Worker’s strategy:** Type $H$ chooses $e = e^*$ and type $L$ chooses $e = 0$. After observing $w_1, w_2$, both types choose the highest offer if $w_1, w_2$ are different, and firm 1 if they are the same.

- **Firms’ belief:** Each firm believes that a worker is type $H$ if he chooses $e = e^*$, and type $L$ otherwise.

- **Firms’ strategies:** Each firm offers the wage $H$ to a worker who chooses $e = e^*$, and $L$ to a worker who chooses any other value of $e$.

Let’s check that the conditions for consistency of beliefs and optimality of strategies are satisfied.

- Consistency of beliefs: take the worker’s strategy as given. The only information sets of the firm that are reached with positive probability are after $e = 0$ and $e = e^*$; at all the rest, the firms’ beliefs may be anything. At the information set after $e = 0$, the only correct belief is $P(H|e = 0) = 0$. At the information set after $e = e^*$, the only correct belief is $P(H|e = e^*) = 1$. So these beliefs are consistent.
• Optimality of firm’s strategy: Each firm’s payoff is 0, given its beliefs and strategy. If a firm deviates by offering a higher wage, it will make a negative profit. If it deviates by offering a lower wage, it gets a payoff of 0 since the worker will choose the other firm. So, there is no incentive to deviate.

• Optimality of worker’s strategy: In the last subgame, the worker’s strategy of choosing the higher wage is clearly optimal. Let’s consider the worker’s choice of e:

  – Type H: If the worker maintains the strategy and chooses \( e = e^* \), he will get a wage offer of \( H \) and his payoff will be \( H - \frac{e^*}{H} \). If the worker deviates and chooses any other \( e \), he will get a wage offer of \( L \) and his payoff will be \( L - \frac{e}{L} \). The highest possible payoff when deviating is when \( e = 0 \), which gives a payoff of \( L \). Therefore, in order for our hypothetical equilibrium to be optimal, we need \( H - \frac{e^*}{H} \geq L \), or

    \[
    e \leq H(H - L)
    \]

  – Type L: If the worker maintains the strategy and chooses \( e = 0 \), he will get a wage offer of \( L \) and his payoff will be \( L \). If the worker deviates and chooses anything but \( e^* \), he still gets a wage offer of \( L \) and a lower payoff of \( L - \frac{e}{L} \). If the worker deviates and chooses \( e^* \) (i.e. imitates a H-type) then he gets a wage offer of \( H \), for a total payoff of \( H - \frac{e^*}{H} \). For our hypothetical equilibrium to be optimal, we need \( L \geq H - \frac{e^*}{H} \) or

    \[
    e^* \geq L(H - L)
    \]

Combining these requirements, the condition for this equilibrium to be optimal is:

    \[
    L(H - L) \leq e^* \leq H(H - L)
    \]

If this is satisfied, then separating equilibria exist in which H-type workers can be distinguished from L-type workers by their choice of \( e \). This is not the only type of equilibrium that exists: there may also exist pooling equilibria, given the same values of \( H \) and \( L \), in which both types of workers choose the same amount of education.