CUR 412: Game Theory and its Applications, Lecture 8
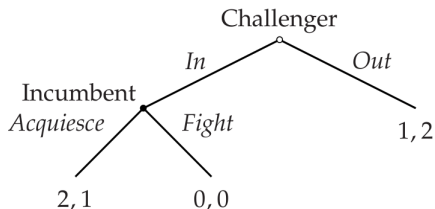
Prof. Ronaldo CARPIO

May 15, 2015

Announcements

- We will reschedule last week's lecture; I will let you know when it is scheduled.
- Midterm: highest=93, mean=74, SD=17.
- The solutions will be posted on the web site later today.
- Also, HW #3 will be posted later today, due in 2 weeks.

- Extensive games can model a situation where players move sequentially.

- We describe extensive games with a *game tree*.

- Each *node* in the tree corresponds to a player's turn to move.

- Each branch from a node corresponds to an action of the player who moved.

- A *history* is a sequence of actions (i.e. a path from the *root node* to some other node).

- A *terminal history* is a path from the root node to a leaf node (i.e. an "ending" to the game).

- Players have preferences over terminal histories.

- A *strategy* of player i specifies which action to play at every node at which it is player i's turn to move.

- A *strategy profile s* is a list of all players' strategies.

- The *outcome* of following the strategy profile s is the terminal history generated by following the actions specified in s.

- The *strategic form* of an extensive game Γ is the strategic game where each player's action set is his set of strategies in Γ.

- We can define Nash equilibria in the usual way for this strategic form.

- However, equilibria may contain *incredible threats*: actions which would be irrational when they are actually played in sequence.

Strategic Form of Entry Game

▸ The strategic form of the Entry Game is:

		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

▸ There are two Nash equilibria: (*In*, *Acquiesce*) and (*Out*, *Fight*).

▸ The first NE is the same as the one found with backwards induction.

▸ In the second NE, the incumbent chooses *Fight*. However, if *In* is taken as given, this is not rational. This is called an *incredible threat*.

▸ If the incumbent could commit to *Fight* at the beginning of the game, it would be credible.

Subgames

- The concept of Nash equilibrium ignores the sequential structure of an extensive game.

- It treats strategies as choices made once and for all at the beginning of the game.

- However, the equilibria of this method may contain incredible threats.

- We'll define a notion of equilibrium that excludes incredible situations.

- Suppose Γ is an extensive form game with perfect information.

- The *subgame* following a non-terminal history h, $\Gamma(h)$, is the game beginning at the point just after h.

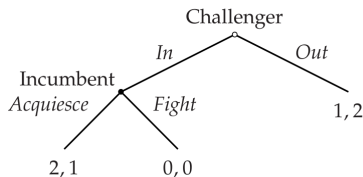- A *proper subgame* is a subgame that is not Γ itself.
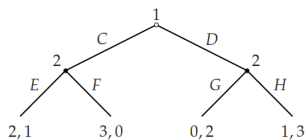
Subgames

▸ This game has two proper subgames:
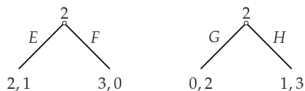
Subgame Perfect Equilibria

- A *subgame perfect equilibrium* is a strategy profile s^* in which each subgame's strategy profile is also a Nash equilibrium.

- Each player's strategy must be optimal for all subgames that have him moving at the beginning, not just the entire game.

- (*Out*, *Fight*) is a NE, but is not a subgame perfect equilibrium because in the subgame following *In*, the strategy *Fight* is not optimal for the incumbent.

Subgame Perfect Equilibria

- Every subgame perfect equilibrium is also a Nash equilibrium, but not vice versa.

- A subgame perfect equilibrium induces a Nash equilibrium in every subgame.

- In games with finite histories, subgame perfect equilibria are consistent with backwards induction.

Backwards Induction in Finite-Horizon Games

- In a game with a *finite horizon* (i.e. finite maximum length of all terminal histories), we can find all SPNE through backwards induction.

- This procedure can be interpreted as reasoning about how players will behave in future situations.

- Procedure:
    - For all subgames of length 1 (i.e. 1 action away from a terminal node), find the optimal actions of the players.
    - Take these actions as given. For all subgames of length 2, find the optimal actions of the players...
    - Repeat until we cover the entire tree.
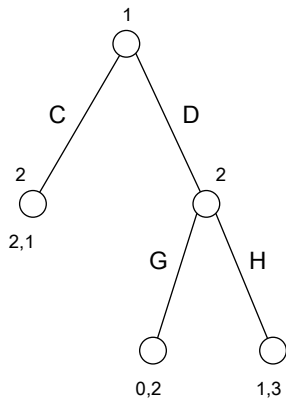
Example 170.1
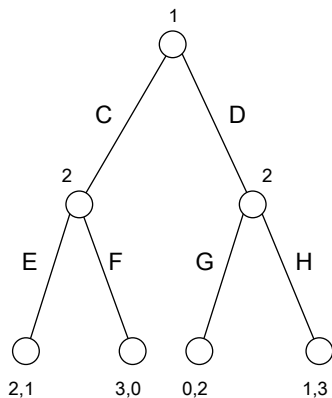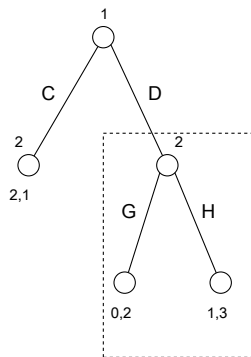

▸ There are 2 subgames with length 1.

Example 170.1
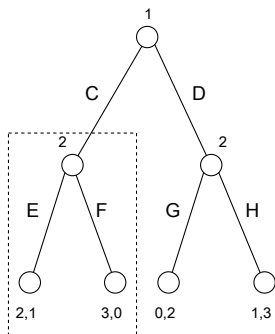
- Consider the left subgame. It is Player 2's turn to move.
- Player 2's optimal action is E, resulting in payoff $(2, 1)$.
- We will assume Player 2 always chooses E, so the payoff of this subgame is $(2, 1)$.
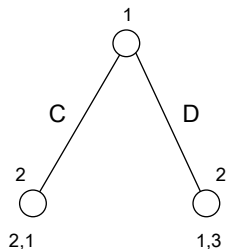
Example 170.1

▸ Therefore, the payoff to Player 1 choosing C is $(2, 1)$.

Example 170.1

- Consider the right subgame. It is Player 2's turn to move.
- Player 2's optimal action is H, resulting in payoff $(1,3)$.
- We will assume Player 2 always chooses H, so the payoff of this subgame is $(1,3)$.
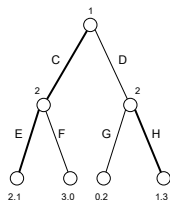
Example 170.1

- Therefore, the payoff to Player 1 choosing D is $(1, 3)$.

- Now, Player 1's optimal action is C.

- Backwards induction gives the strategy pair (C, EH).

- The outcome of (C, EH) is the terminal history CE with payoff $(2, 1)$.

Example 170.1
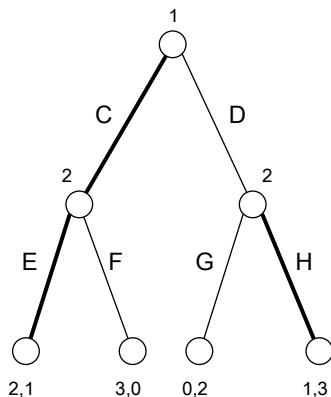
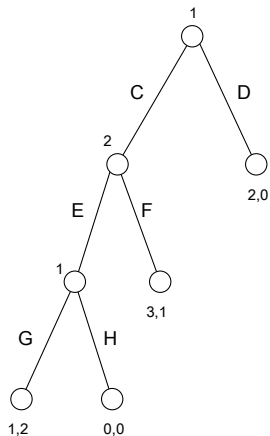▸ We mark the optimal actions at each node with thick lines.

Strategic Form of 170.1

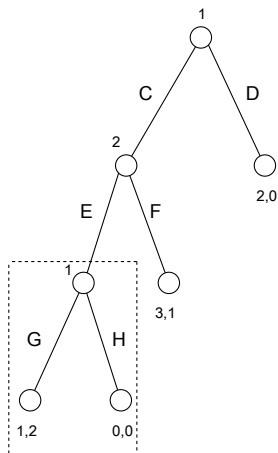|	EG	EH	FG	FH
C	2,1	2,1	3,0	3,0
D	0,2	1,3	0,2	1,3

- Let's compare the backwards induction result (C, EH) to the NE of the strategic form.
- (C, EG) and (C, EH) are NE of strategic form.
- However, (C, EG) includes a non-optimal action for Player 2 in the right subgame, so is not a subgame-perfect NE.
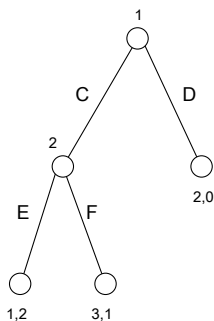
Example 160.1

- There is one subgame with length 1, and one subgame with length 2.
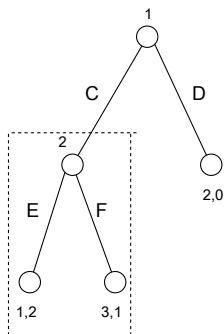
Example 160.1

- In this subgame, it is Player 1's turn to move.
- Optimal action is G, resulting in payoff $(1, 2)$.

Example 160.1
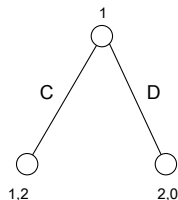

- Assume Player 1 chooses G with certainty.
- Then, the payoff to choosing E is $(1, 2)$.

Example 160.1
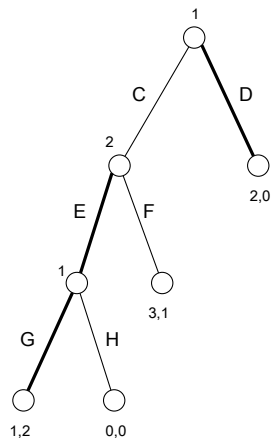

- In this subgame, it is Player 2's turn to move.
- Optimal action is E, resulting in payoff $(1, 2)$.

Example 160.1

- Optimal action is D, resulting in payoff $(2, 0)$.
- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.

Example 160.1

- Backwards induction gives the strategy pair (DG, E) resulting in terminal history D.
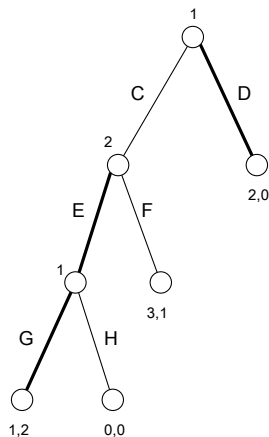
	E	F
CG	1,2	3,1
CH	0,0	3,1
DG	2,0	2,0
DH	2,0	2,0

- NE of strategic form are: $(CH, F), (DG, E), (DH, E)$.

- Only (DG, E) is a subgame perfect NE.

Example 158.1 (Variant of Entry Game)
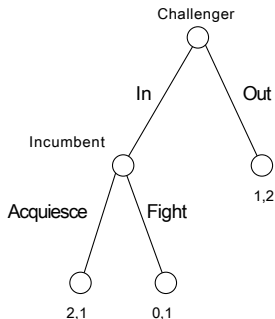
- What if there are multiple optimal actions in a subgame? Then we need to keep track of them separately.

- This is a variant of the entry game in which the Incumbent is indifferent between *Acquiesce*, *Fight*.

Example 158.1 (Variant of Entry Game)
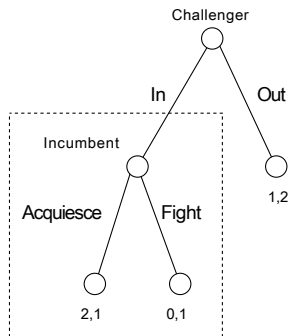
- In this subgame, both *Acquiesce* and *Fight* are optimal actions.
- We cannot eliminate either as an irrational choice. So, we keep track of both possibilities.

Example 158.1 (Variant of Entry Game)

assuming Acquiesce is chosen   assuming Fight is chosen
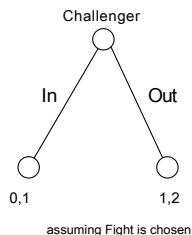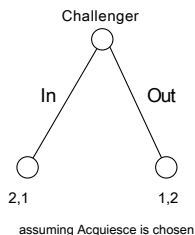
- ▸ Backwards induction gives (*In, Acquiesce*) and (*Out, Fight*).
- ▸ In this case, the NE of the strategic form are the same as the subgame-perfect NE.

Continuous Action Sets

▸ The action set at a node may be infinite (e.g. if the player chooses a real number).

▸ In this case, we graphically represent this with an arc between the lowest and highest possible values.

▸ Effectively, there are an *infinite* number of branches in the game tree at this node.

▸ Suppose it is Player i's turn to move after all of these branches. Then Player i's strategy profile must specify an action for *all* possible branches.
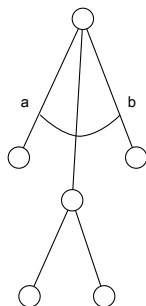
- If the infinite set of actions is an interval of real numbers $[a, b]$, then Player i's strategy profile for this node must be a *function* over $[a, b]$.

- For a strategy profile to be a subgame perfect NE, it must induce a NE at each of the infinite subgames.

- ▸ A union and a firm are bargaining.
- ▸ First, the union presents a wage demand $w \geq 0$.
- ▸ The firm chooses an amount $L \geq 0$ of labor to hire.
- ▸ The firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labor, and 2500 if $L > 50$.
- ▸ The price of output is 1.
- ▸ The firm's preferences are represented by its profits.
- ▸ The union's preferences are represented by the total wage bill, wL.

Exercise 177.1 (Firm-Union Bargaining)
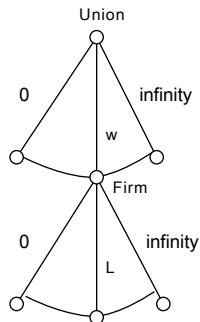
- The firm's payoff is its profit, given by:

$$\Pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- Union's payoff: wL

Exercise 177.1 (Firm-Union Bargaining)

$$\Pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50 \end{cases}$$

- For every $w \geq 0$, there is a subgame where the firm's payoff depends on w.

- Profit has a quadratic part (if $L \leq 50$) and a linear part (if $L > 50$), and is continuous at $L = 50$.

- We want to find the profit-maximizing choice of L.

- The linear part is decreasing in L, so we can ignore it (its maximum is at $L = 50$).

- Quadratic part is maximized at $L^* = \frac{100 - w}{2}$.

Exercise 177.1 (Firm-Union Bargaining)

- Quadratic part is maximized at $L^* = \frac{100-w}{2}$.
- Firm's profit is:

$$\frac{100-w}{2}\left(100 - \frac{100-w}{2}\right) - w\frac{100-w}{2} = \frac{(w-100)^2}{4}$$

- Profit is always non-negative. Firm's best response correspondence is:

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

$$B_f(w) = \begin{cases} L \geq 50 & \text{if } w = 0 \\ L = \frac{100-w}{2} & \text{if } 0 < w \leq 100 \\ L = 0 & \text{if } w > 100 \end{cases}$$

▸ Now, consider the union's decision.

▸ If $w = 0$ or $w > 100$, union's payoff is 0.

▸ $wB_f(w) = \frac{w(100-w)}{2}$ is maximized at $w^* = 50$.

▸ $L^* = B_f(50) = 25$.

- The set of subgame perfect NE is:

- Union's strategy profile: at the empty history, choose $w = 50$.

- Firm's strategy profile: at the subgame following the history w, choose an element of $B_f(w)$.

- Note that the firm has an infinite number of strategy profiles, but there is only one equilibrium outcome, since only the subgame after $w = 50$ will be realized.

- Firm's payoff is 625 and union's payoff is 1250.

- Is there an outcome that both players prefer to the SPNE outcome with payoffs $(1250, 625)$?

- Suppose that instead of each player maximizing his own payoff, a *social planner* could choose both w and L.

- The sum of payoffs is $L(100 - L) - wL + wL = L(100 - L)$ which is maximized at $L = 50$. The choice of w then allocates payoffs to the firm and union.

- For example, if $w = 30$, then the firm's payoff is 1000 and the union's payoff is 1500.

- This is an illustration that individual maximization may not achieve the most efficient outcome.

- Is there a Nash equilibrium outcome that differs from any subgame perfect NE outcome?

- Suppose the union's strategy is: offer $w = 100$ and the firm's strategy profile is: for any w, offer $L = 0$.

- The firm has no incentive to deviate, since it will make a negative payoff for any $L > 0$.

- The union has no incentive to deviate, because it will get a payoff of 0 for any choice of w.

- This is not subgame perfect, since the firm's strategy is not optimal for $w < 100$.

▸ **Proposition**: In a finite horizon extensive game with perfect information, the set of strategy profiles isolated by backwards induction is the set of all subgame-perfect equilibria.

▸ **Proposition**: Every finite extensive game with perfect information has a subgame perfect equilibrium.

Announcements

- ▸ Please read Chapter 6 for next meeting.
- ▸ We will reschedule last week's lecture; I will let you know when it is scheduled.
- ▸ Midterm solutions will be posted on the web site later today.
- ▸ Also, HW #3 will be posted later today, due in 2 weeks.