Appendix to Ch. 4

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- Suppose there are two goods to choose from. Let X, Y be the amounts of each good in the market basket.
- As before, assume that the consumer's preferences are described by a *utility function* $U(X, Y)$, with the following properties:
  - $U(X, Y)$ is continuous and differentiable.
  - $U(X, Y)$ is increasing in both X and Y.
  - For any utility level u, the set of bundles X, Y that give utility $U(X, Y) \geq u$ forms a convex set.
- Prices for the two goods are P_x, P_y respectively.
- Consumer's income is I.

Constrained Optimization

- We can formulate the consumer choice problem as a *constrained optimization* problem:

$$\max_{X,Y} U(X,Y) \qquad \text{subject to} \qquad P_x X + P_y Y = I$$

- $U(X,Y)$ is the *objective function*, and $P_x X + P_y Y = I$ is the *constraint*
- To find the solution to this problem, we can use the method of *Lagrange multipliers*, which goes as follows:
- Rewrite the budget constraint as: $P_x X + P_y Y - I = 0$
- Form the *Lagrangian equation*, which is simply the objective function $U(X,Y)$ minus the constraint multiplied by a variable λ.

$$L(X,Y,\lambda) = U(X,Y) - \lambda(P_x X + P_y Y - I)$$

- λ is called a *Lagrange multiplier*.

Constrained Optimization

$$L(X,Y,\lambda) = U(X,Y) - \lambda(P_x X + P_y Y - I)$$

- We want to maximize this equation with respect to X, Y, and λ.
- Write down the partial derivatives with respect to each variable and set it to 0:

$$\frac{\partial L}{\partial X} = \frac{\partial U}{\partial X} - \lambda P_x = 0$$

$$\frac{\partial L}{\partial Y} = \frac{\partial U}{\partial Y} - \lambda P_y = 0$$

$$\frac{\partial L}{\partial \lambda} = P_x X + P_y Y - I = 0$$

- Note that the third equation is simply the budget constraint.
- $\frac{\partial U(X,Y)}{\partial x}$ is the *marginal utility* with respect to X.

Constrained Optimization

$$\frac{\partial L}{\partial X} = \frac{\partial U}{\partial X} - \lambda P_x = 0$$
$$\frac{\partial L}{\partial Y} = \frac{\partial U}{\partial Y} - \lambda P_y = 0$$
$$\frac{\partial L}{\partial \lambda} = P_x X + P_y Y - I = 0$$

- Combining the first two equations, we get the $MRS = price\ ratio$ condition for optimality:

$$\frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \frac{P_x}{P_y}$$

- We can rearrange this to get the *equal marginal principle*:

$$\lambda = \frac{\frac{\partial U}{\partial X}}{P_x} = \frac{\frac{\partial U}{\partial Y}}{P_y}$$

Consumer Demand

- For certain forms of the utility function $U(X, Y)$, we can explicitly compute the formula for demand.
- Suppose the utility function is: $U(X, Y) = X^{\frac{1}{2}} Y^{\frac{1}{2}}$.
- The marginal utility functions are:

$$\frac{\partial U}{\partial X} = \frac{1}{2} X^{-\frac{1}{2}} Y^{\frac{1}{2}}$$

$$\frac{\partial U}{\partial Y} = \frac{1}{2} X^{\frac{1}{2}} Y^{-\frac{1}{2}}$$

$$MRS = \frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \frac{Y}{X} = \frac{P_x}{P_y}$$

Consumer Demand

$$MRS = \frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \frac{Y}{X} = \frac{P_x}{P_y}$$

- We can rearrange this to give $P_x X = P_y Y$.
- From the budget equation, we have $P_x X + P_y Y = I$. Substituting in the previous equation, we get

$$P_x X = P_y Y = \frac{I}{2}$$

- which gives us the demand for X and Y in terms of P_x, P_y and I:

$$X = \frac{I}{2P_x}, Y = \frac{I}{2P_y}$$